5. (a) If $I_n = \frac{d^n}{dx^n}(x^n \log x)$, prove that $I_n = nI_{n-1} + (n-1)!$ and hence prove that

$$I_n = n!\left[\frac{1}{n} + \frac{1}{n-1} + \frac{1}{n-2} + ... + \frac{1}{2} + 1 + \log x\right].$$

(b) If $y = e^{a\sin^{-1}x}$, prove that

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+a^2)y_n = 0.$$

(c) State the Taylor's theorem for a function defined on some open interval I of real numbers. Also, find the Taylor's series expansion of e^{2x}, for $x \in \mathbb{R}$ about zero.

(d) Sketch a graph of the equation $y = \frac{2x^2-8}{x^2-16}$.

6. (a) Find $\lim_{x\to 0}(1+\sin x)^{\frac{1}{x}}$.

(b) Find the asymptotes of the rational function, $y = \frac{x^3 - x^2 - 8}{x-1}$

(c) Examine the nature of origin on the curve.

$$(2x+y)^2 - 6xy(2x+y) - 7x^3 = 0$$

(d) Trace the curve $r = 1 - \cos\theta$.

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This question paper contains 4 printed pages]

Your Roll No. : ....

Sl. No. of Q. Paper : **2058** I

Unique Paper Code : 2352011202

Name of the Paper : DSC-5: Calculus

Name of the Course : **B.Sc. (H) Mathematics (NEP-UGCF 2022)**

Semester : II

Time : 3 Hours ***Maximum Marks : 90***

Instructions for Candidates :

(a) Write your Roll No. on the top immediately on receipt of this question paper.

(b) Attempt **all** questions by selecting **three** parts from each question.

(c) The parts of each question are to be attempted together.

(d) **All** questions carry equal marks.

(e) Use of Calculator is **not** allowed.

1. (a) Let $f: A \to R$ and let c be a cluster point of A. Show that $\lim_{x\to c} f = L$ if and only if for every sequence $\langle x_n \rangle$ in A that converges to c such that $x_n \neq c$ for all $n \in N$, the sequence $\langle f(x_n) \rangle$ converges to L.

(b) Show that $\lim_{x\to 0}\sin\frac{1}{x}$ does not exist in $\mathbb{R}$.

(c) Using $\epsilon-\delta$ definition, show that $\lim_{x\to 1}\frac{x}{1+x}=\frac{1}{2}$.

(d) Find $\lim_{x\to 0}\frac{\sqrt{1+2x}-\sqrt{1+3x}}{x+2x^2}$ where $x > 0$.

2. (a) Suppose that the functions f and g have limits in $\mathbb{R}$ as $x\to\infty$ and that $f(x)\le g(x)$ for all $x\in(0,\infty)$. Prove that $\lim_{x\to\infty} f\le \lim_{x\to\infty} g$.

(b) Prove by verifying the $\epsilon-\delta$ property that the function $f(x) = x^2$ is continuous at $x_0 = 2$.

(c) State the Intermediate Value Theorem and show that $x = \cos x$ for some x in $\left(0,\frac{\pi}{2}\right)$.

(d) Show that the function $f(x)=\frac{1}{x^2}$ is not uniformly continuous on $(0,1)$.

3. (a) Prove that every uniformly continuous function on a set A is continuous on A. Is the converse true? Justify.

(b) If f and g are functions that are differentiable at the point a, then prove that $(fg)'(a)=f(a)g'(a)+f'(a)g(a)$.

(c) Let $f(x)=|x|+|x+1|$ for $x\in\mathbb{R}$. Determine the set of points at which the function f is not differentiable. Justify your answer.

(d) Prove $|\cos x-\cos y|\le|x-y|$ for all $x,y\in\mathbb{R}$.

4. (a) Suppose that $f:[0,2]\to\mathbb{R}$ is continuous on $[0,2]$, differentiable on $(0, 2)$, and that $f(0) = 0$, $f(1) = 1$, $f(2) = 1$.

(i) Show that $f'(c_1)=1$ for some $x\in(0,1)$.

(ii) Show that $f'(c_2)=0$ for some $x\in(1,2)$.

(iii) Show that $f'(c)=\frac{1}{10}$ for some $x\in(0,2)$.

(b) State and prove Mean Value Theorem.

(c) Show that $e^x\ge 1+x$ for all $x\in\mathbb{R}$.

(d) Prove that if f and g are differentiable on $\mathbb{R}$, if $f(0)=g(0)$ and if $f'(x)\le g'(x)$ for all $x\in\mathbb{R}$, then $f(x)\le g(x)$ for all $x\ge 0$.