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Your Roll No.....

Sr. No. of Question Paper : 1772

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Unique Paper Code : 2352013603

Name of the Paper : DSC-18 Complex Analysis

Name of the Course : **Bachelor of Science (Honours Course)
Mathematics**

Semester : VI

Duration : 3 Hours

Maximum Marks : 90

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. **All** questions are compulsory.
2. **All** questions carry equal marks.
3. Attempt **two** parts from each question.

1. (a) (i) Solve $e^z = 3$, where z is any complex number.

(ii) Find the values of $\text{Log}(1 + i)^2$ and $\text{Log}(-1 + i)^2$ and justify the truth of $\text{Log}(z_1 z_2) = \text{Log}(z_1) + \text{Log}(z_2)$ for $z_1, z_2 \in \mathbb{C}$.

- (b) Prove that if $f'(z) = 0$ everywhere in a domain D , then $f(z)$ must be constant throughout D .

- (c) Show that each of the following functions is nowhere analytic :

(i) $f(z) = xy + iy$

(ii) $f(z) = 2xy + i(x^2 - y^2)$

2. (a) Prove that if the function $f(z) = u(x, y) + iv(x, y)$ be defined throughout some ϵ -neighbourhood of a point $z_0 = x_0 + iy_0$, and the following conditions hold true :

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- (i) The first order derivatives of the functions u and v with respect to x and y exist everywhere in the neighbourhood;
- (ii) Those partial derivatives are continuous at (x_0, y_0) and satisfy the Cauchy- Riemann equations at (x_0, y_0) ,

Then $f'(z)$ exists and $f'(z) = u_x + iv_x$ at $z = z_0$.

- (b) Find the image of the domain $x > 0, y > 0, xy < 0$ under the mapping $w = z^2$.
- (c) If z_0 and w_0 are points in the z and w planes, respectively, then prove that

$$(i) \lim_{z \rightarrow \infty} f(z) = w_0 \text{ if and only if } \lim_{z \rightarrow 0} f\left(\frac{1}{z}\right) = w_0.$$

$$(ii) \lim_{z \rightarrow \infty} f(z) = \infty \text{ if and only if } \lim_{z \rightarrow 0} \frac{1}{f\left(\frac{1}{z}\right)} = 0.$$

- 3. (a) Prove that if $w(t)$ is piecewise continuous complex-valued function defined on an interval $a \leq t \leq b$, the :

$$\left| \int_a^b w(z) dt \right| \leq \int_a^b |w(t)| dt.$$

Find an upper bound for $\left| \int_C \frac{z+4}{z^3-1} dz \right|$, where C is the arc of the circle $|z| = 2$ from $z = 2$ to $z = 2i$.

- (b) Suppose that a function $f(z)$ is continuous in a domain D . Prove that if $f(z)$ has an antiderivative $F(z)$ throughout D , then the value of the integral of $f(z)$ along contours lying entire y in D and extending from any fixed point z_1 to any other fixed point z_2 is

$$\int_{z_1}^{z_2} f(z) dz = F(z_2) - F(z_1)$$

Use it to find the value if $\int_0^{\pi+2i} \cos\left(\frac{z}{2}\right) dz$.

- (c) State Cauchy-Goursat theorem. Use it to show that

$$\int_C f(z) dz = 0, \text{ when}$$

(i) $f(z) = \frac{1}{z^2 + 2z + 2}$ and C is the unit circle $|z| = 1$.

(ii) $f(z) = \frac{5z + 7}{z^2 - 4z + 3}$ and C is the unit circle $|z - 2| = 2$.¹

4. (a) For a function f given to be analytic inside and on a positively oriented circle C_R , centered at z_0 and with radius R , if M_R denotes the maximum value of $|f(z)|$ on C_R , then prove that $|f^n(z_0)| \leq \frac{n! M_R}{R^n}$.

Hence, show that if f is an entire function such that $|f(z)| \leq A|z|$ for all z , where A is a fixed positive number, then $f(z) = a_1 z$, where a_1 is a complex constant.

- (b) Evaluate $\int_C \frac{3z+1}{z(z-2)^2} dz$, where C is the figure-eight contour formed by two circles with centres at $(0,0)$ and $(2,0)$ respectively with radii 0.5 and 1.5 respectively, described in the positive direction,

- (c) State and prove Liouville's theorem. Apply this to show that if $f(z)$ is entire and the function $u(x, y) = \operatorname{Re}[f(z)]$ is bounded above by u_0 at all points (x, y) in the xy plane, then $u(x, y)$ must be constant throughout the plane.

5. (a) Give all the Laurent series expansions in the powers of z for the function

$$f(z) = \frac{1}{z^2(1-z)}$$

and specify the domains in which those expansions are valid.

- (b) Suppose that $z_n = x_n + iy_n$ ($n = 1, 2, 3, \dots$) and $S = X + iY$. Then, prove that $\sum_{n=1}^{\infty} z_n = S$ iff $\sum_{n=1}^{\infty} x_n = X$ and $\sum_{n=1}^{\infty} y_n = Y$. Also show that if $\sum_{n=1}^{\infty} z_n = S$ then $\sum_{n=1}^{\infty} \overline{z_n} = \overline{S}$.

- (c) State Taylor's theorem and derive Taylor's series representations for the following :

$$(i) \frac{z}{(z^4 + 9)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{3^{2n+2}} z^{4n+1} \text{ where, } |z| < \sqrt{3}$$

$$(ii) \frac{\sin(z^2)}{z^4} = \frac{1}{z^2} - \frac{z^2}{3!} + \frac{z^6}{5!} - \frac{z^{10}}{7!} \dots\dots\dots, \text{ when } z \neq 0.$$

6. (a) State Cauchy's residue theorem and use it to evaluate the integral

$$\int \frac{dz}{z^3(z+4)}$$

over the circle $|z + 2| = 3$ oriented anticlockwise.

- (b) Define the terms: isolated and non-isolated singular points. Write the principal part of each of the following functions at their respective isolated singular points and determine whether that point is a pole, a removable singular point, or an essential singular point.

$$(i) z \exp\left(\frac{1}{z}\right)$$

$$(ii) \frac{1 - \cos z}{z^2}$$

$$(iii) \frac{1}{(2-z)^3}.$$

- (c) Use residues to evaluate $\int_0^{2\pi} \frac{d\theta}{5 + 4\sin\theta}.$