(b) show that

$$\sum_{n=0}^{\infty}(-1)^n x^{2n}=\frac{1}{1+x^2} \text{ for } x\in(-1,1)$$

(c) let $f(x)=\sum_{n=0}^{\infty}a_n x^n$ be a power series with finite positive radius of convergence R. Prove that if the series converge at x = R, then f is continuous at x = R. If the series converges at x = – R, then f is continuous at x = –R

6. (a) Let f be a continuous function on [a,b]. Show that there exists a sequence (p_n) of polynomials such that $p_n \to f$ uniformly on [a,b] and $p_n(a) = f(a)$ for all n.

(b) If $x \geq 0$ and $x \in \mathbb{N}$, show that

$$\frac{1}{1+x}=1-x+x^2-x^3+\ldots\ldots\ldots\ldots+(-x)^{n-1}+\frac{(-x)^n}{1+x}$$

Hence show that

$$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\ldots\ldots\ldots+(-1)^n\frac{x^n}{n}+\int_0^x\frac{(-t)^n}{1+t}dt$$

(c) Show that if x > 0 then

$$1-\frac{x^2}{2}+\frac{x^4}{24}-\frac{x^6}{720}\leq\cos x\leq1-\frac{x^2}{2}+\frac{x^4}{24}$$

Use this inequality to establish a lower bound for π .

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This question paper contains 4 printed pages]

Your Roll No. : ....

Sl. No. of Q. Paper : **1813** **I**

Unique Paper Code : 2352012401

Name of the Paper : Sequences and Series of Functions (DSC)

Name of the Course : **B.A/B.Sc.(H)**

Semester : IV

Time : 3 Hours ***Maximum Marks : 90***

Instructions for Candidates :

(a) Write your Roll No. on the top immediately on receipt of this question paper.

(b) Attempt **all** questions by selecting any **two** parts from each question.

(c) **All** questions carry equal marks.

(d) Use of calculator is **NOT** allowed.

1. (a) State and prove Cauchy's criterion for uniform convergence for sequence of functions.

(b) Show that the sequence $(n^2x^2e^{-nx})$ converges uniformly on the interval $[1,\infty)$, but that it does not converge uniformly in $[0,\infty)$.

(c) (i) Show that if the sequences (f_n), (g_n) converge uniformly on the set A to f and g, respectively, then the sequence (f_n+g_n) converges uniformly to f + g .

(ii) Discuss the uniform convergence of the sequence of functions $\left(\frac{x^2+nx}{n}\right)$ on $\mathbb{R}$.

P.T.O.

2. (a) Let (f_n) be a sequence of continuous functions on a set $A \subseteq \mathbb{R}$ and suppose that (f_n) converges uniformly on A to a function $f : A \to \mathbb{R}$. Prove that f is continuous on A.

(b) Let $f_n(x) = \dfrac{nx}{1+nx^2}$ for $x \in A = [0,\infty)$. Show that each f_n is bounded on A, but the pointwise limit f of the sequence (f_n) is not bounded on A. Does the sequence (f_n) converge uniformly to f on A?

(c) (i) Let $f_n(x) = \dfrac{x^n}{n}$, for $x \in [0,1]$. show that the sequence (f_n) of differentiable functions converges uniformly to a differentiable function f on [0,1], and that the sequence (f'_n) converges on [0,1], but that $g(1) \neq f'(1)$.

(ii) If a>0, show that $\lim\limits_{n\to\infty}\int_a^{\pi} \dfrac{\sin nx}{nx}dx = 0$.

3. (a) State and prove Weierstrass M-test for uniform convergence of the series of functions. Hence, show that $\sum b_n \sin nx$ is a uniformly convergent series given that $\sum b_n$ is an absolutely convergent series.

(b) Let f_n be a real valued function on [a,b] that has derivative f_n' on [a, b] for each $n \in \mathbb{N}$. Suppose that the series' $\sum f_n$ converges for at least one point of [a,b] and the series of derivatives $\sum f_n'$ converges uniformly on [a,b]. Show that there

exists a real valued function f on [a,b] such that $\sum f_n$ converges uniformly to f on [a,b]. Also, that f has a derivative on [a,b] and $f' = \sum f_n'$

(c) Discuss the pointwise convergence of $\sum_{n=1}^{\infty} \dfrac{1}{x^n + 1}, x \geq 0$. Also, show that the given series is uniformly convergent on [2, ∞) but not uniformly convergent on $(1, \infty)$.

4. (a) Prove that $\sum_{n=1}^{\infty} \sin\left(\dfrac{x}{n^2}\right)$ converges for all $x \in \mathbb{R}$. Also, show that the series converges uniformly on [–a, a], a>0 but convergence is not uniform on $\mathbb{R}$.

(b) Let (f_n) be a sequence of integrable functions on [a,b] and suppose that $\sum_{n=1}^{\infty} f_n$ converges uniformly to f on [a,b]. Show that f is integrable on [a,b] and $\int_a^b f = \sum\limits_{n=1}^{\infty}\int_a^b f_n$

(c) Prove that $\sum_{n=1}^{\infty} \dfrac{1}{n^2 x^2}$ is convergent for all $x \neq 0$. Also, show that the convergence is not uniform for $|x| < 1$ but is uniform for $|x| \geq 1$.

5. (a) Define power series and its radius of convergence. Evaluate the radius of convergence for the series $\sum \dfrac{n!^2}{(2n)!} x^n$

P.T.O.