(c) Use Monte Carlo simulation to approximate the probability of heads occurring when a fair coin is tossed n times.

[This question paper contains 8 printed pages.]

Your Roll No..............

Sr. No. of Question Paper : 5783 **L**

Unique Paper Code : 2354000010

Name of the Paper : Introduction to Mathematical Modeling (GE)(NEP-UGCF)

Name of the Course : **Common Prog Group**

Semester : VI

Duration : 3 Hours Maximum Marks : 90

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. All **six** questions are compulsory.
3. Attempt any **two** parts from each question.
4. **All** questions carry equal marks.
5. Use of non-programmable Scientific Calculator is allowed.

1. (a) A radioactive substance decays according to the model

$$\frac{dN}{dt} = -kN, \qquad N(t_0) = n_0$$

(i) Solve the above initial value problem.

(ii) If the half-life of the substance is τ, find k in terms of τ.

(b) Consider the lake pollution model governed by the differential equation :

$$\frac{dC}{dt} = \frac{F}{V}c_{in} - \frac{F}{V}C$$

where C(t) is the concentration of pollutant in the lake at time t, F is the flow rate, V is the volume of the lake, and C_{in} is the concentration of pollutant in the inflowing water.

(i) Solve the above differential equation subject to the initial condition $C(0) = c_0$.

5. (a) For the following data, formulate the mathematical model minimizes the largest deviation between the data and the model $y = c_1x^2 + c_2x + c_3$:

x	0.1	0.2	0.3	0.4	0.5
y	0.06	0.12	0.36	0.65	0.95

(b) Use the middle-square method to generate 10 random numbers using seed $x_0 = 653217$.

(c) Write a Monte Carlo algorithm to approximate the volume trapped between the two paraboloids $z = 8 - x^2 - y^2$ and $z = x^2 + 3y^2$, given that they intersect on the cylinder $x^2 + 2y^2 = 4$.

6. (a) Fit the curve $y = ax^2$ by using the Least Squares Method for the data :

x	5	10	15	20	25
y	2	4	7	12	20

(b) Generate ten random numbers using the Linear Congruence Method,

$$x_i \equiv (ax_{i-1} + b) \bmod (m)$$

where $m = 16$, $a = 5$, and $b = 3$, with seed $x_0 = 3$.

4. (a) For the battle model, determine the appropriate input-output compartmental model and associated differential equations for the number of soldiers in both the red and blue armies, if both the armies use aimed fire. Modify the model if blue army uses random fire.

(b) For the predator-prey model with density dependence for the prey and DDT acting on both species

$$\frac{dX}{dt} = \beta_1 X\left(1 - \frac{X}{K}\right) - c_1XY - p_1X, \quad \frac{dY}{dt} = c_2XY - \alpha_2 Y - p_2Y,$$

Show that (X, Y) is one equilibrium point, where

$X = \frac{\alpha_2 + p_2}{c_2}$, $Y = \frac{\beta_1\left(1 - \frac{X_e}{K}\right) - p_1}{c_1}$ and determine whether there are any other equilibrium points.

(c) Consider a disease where all those infected remain contagious for life. Ignore all births and deaths. Write down suitable word equations for the rate of change of the number of susceptible and infectives. Hence, develop a pair of differential equations.

(ii) How long will it take for the lake's pollution level to reach 5% of its initial level if only fresh water flows into the lake?

(c) A radioactive sample initially contains n_0 nuclei. After 3 hours, the amount reduces to half of its initial value.

(i) Write and solve the differential equation describing the decay model.

(ii) Determine the decay constant k.

(iii) Find the amount remaining after 9 hours.

2. (a) A population changes due to births and deaths, where the per capita birth rate is β and the per capita death rate is α.

(i) Using the balance law, write the corresponding word equation for the rate of change of the population and hence formulate the differential equation governing the population X(t).

(ii) Solve the differential equation obtained, given that $X(0) = x_0$.

(iii) Find an expression for the time required for the population to double in size.

(b) Solve the logistic differential equation $\frac{dX}{dt} = rX(1 - \frac{X}{K})$, given the initial condition $X(0) = x_0$.

(c) In a fish farm, fish are harvested at a constant rate of 2,100 fish per week. The per- capita death rate for the fish is 0.2 fish per day per fish, and the per-capita birth rate is 0.7 fish per day per fish.

(i) Write a word equation describing the rate of change of the fish population. Hence obtain a differential equation for the number of fish. (Define any symbols you introduce.

(ii) If the fish population at a given time is 240,000, give an estimate of the number of fish born in one week.

3. (a) When considering a disease,

(i) Construct a compartmental diagram for the model and develop appropriate word equations for the rates of change of susceptible, contagious infectives and recovered population.

(ii) Using (i), formulate differential equations.

(b) (i) Find the equilibrium solutions of the equations

$$\frac{dX}{dt} = 2X - XY, \frac{dY}{dt} = Y - XY.$$

(ii) Find a relation between Y and X by the chain rule for the differential equations

$$\frac{dX}{dt} = -XY, \frac{dY}{dt} = -2Y.$$

(c) Determine the predator and prey model in the limiting cases of prey with no predator and predator with no prey.