[This question paper contains 2 printed pages.]

Your Roll No................

Sr. No. of Question Paper : 5782 **L**

Unique Paper Code : 2354000009

Name of the Paper : Abstract Algebra

Name of the Course : **COMMON PROG GROUP**

Semester : VI

Duration : 3 Hours Maximum Marks : 90

<u>Instructions for Candidates</u>

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Attempt all questions by selecting **two** parts from each question.
3. **All** parts of a question are to be attempted together.
4. Each part carries **7.5** marks.
5. Use of Calculator not allowed.

1. (a) Define the group GL(2,F), where F is any field. In GL(2, $\mathbb{Z}_{13}$), find the inverse of $\begin{pmatrix} 7 & 4 \\ 1 & 5 \end{pmatrix}$.

   (b) Prove that a group G is Abelian iff $(ab)^{-1} = a^{-1}b^{-1} \quad \forall\ a, b \in G$.

   (c) Define the group U(n), for $n \in \mathbb{N}$ Construct the Cayley table for U(20). Find the inverse of each element of U(20).

2. (a) Let G be an Abelian group under multiplication with identity e. Prove that the subset $H = \{x^2 \mid x \in G\}$ is a subgroup of G.

   (b) Let G be a group. Define the Centralizer, C(a), of an element a in G. Prove that $C(a) = C(a^{-1}) \quad \forall\ a \in G$.

   (c) Define the order of an element in a group G. Find the order of each element in the group U(15).

3. (a) Define a cyclic group. Show that U(14) = <3> = <5>. Is U(14) = <11>?

   (b) Define the order of a permutation. Determine whether the following permutations are even or odd :

   (i) (1243)(3521)

   (ii) (12)(134)(152)

   (c) State & prove Lagrange's theorem for groups.

4. (a) Prove that a quotient group of a cyclic group is cyclic.

   (b) Define a normal subgroup of a group. Let

$$H = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : \ a, b, d \in \mathbb{R}, ad \neq 0 \right\}$$

   Is H a normal subgroup of the group GL(2, $\mathbb{R}$)?

   (c) Show that map $\phi: \mathbb{R}^* \to \mathbb{R}^*$ defined by $\phi(x) = |x|$ is a group homomorphism. Also show that Ker ϕ is a cyclic group.

5. (a) (i) Prove that the set $A = \{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a, b \in \mathbb{Z}\}$ is a subring of the ring R of 2×2 matrices over $\mathbb{Z}$.

   (ii) Prove that the ring $\mathbb{Z}_p$ of integers modulo p, where p is a prime number, is an integral domain.

   (b) Prove that intersection of any collection of subrings of a ring R is a subring of R. Does the result hold for union of subrings ? Justify.

   (c) Show that the set $\mathbb{Q}[\sqrt{2}] = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$ is a commutative ring with unity.

6. (a) Define an ideal of a ring. State the ideal test. Hence prove that $n\mathbb{Z}$ is an ideal of $\mathbb{Z}$, where $n \in \mathbb{Z}$.

   (b) Let f: R $\to$ S be a ring homomorphisms. Prove that

   (i) f(A) is a subring S, where A is a subring of R

   (ii) f(B) = {r $\in$ R : f(r) $\in$ B] is an ideal of R, where B is an ideal of S.

   (c) Prove that the only ideals of a field F are {0} and F itself.