[This question paper contains 4 printed pages.]

Your Roll No................

Sr. No. of Question Paper : 5781 **L**

Unique Paper Code : 2353200011

Name of the Paper : Introduction to Mathematical Modeling

Name of the Course : **B.A. (P) / B.Sc. (Physical Science & Mathematical Science) – DSE**

Semester : VI

Duration : 3 Hours Maximum Marks : 90

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Attempt **all** question by selecting **two** parts from each question.
3. **All** questions carry equal marks.
4. Use of Scientific Calculator is allowed.

1. (a) Consider the problem of pollution in the Lake Burley Griffin in Australia. Assume the lake has a constant volume.

(i). Write down a differential equation describing the concentration of pollution, using V for the volume of the lake, F for flow, $C(t)$ for concentration at time t and C_{in} for the concentration of pollution entering the lake.

(ii).For $V = 28 \times 10^6 m^3, F = 4 \times 10^6 m^3/month$, find how long would it take for the lake with pollution concentration of $10^7 parts/m^3$ to drop below the safety threshold $(4 \times 10^6\ m^3)$ if only fresh water enters the lake. (7.5)

(b) Consider the following differential equation:

$$\frac{dy}{dt} = y - 1$$

Answer the followings:

(i). Find the equilibrium solutions.

(ii). Identify the stable and unstable equilibrium solutions. (7.5)

(c) A population of a certain country grows at a rate of 5% per year. If the initial population is 100, in how much time the population will double. (7.5)

2. (a) Suppose that the population of a country is modelled using the differential equation

$$\frac{dN}{dt} = 0.4N - 0.002X^2,$$

With initial population equals to 200. The unite of time is in month. Find the population of the country after 2 months. (7.5)

(b) Many animal population have decreasing per-capita birth rates when the population density increases, as well as increasing per-capita death rates. Suppose the density-dependent per-capita birth rate $B(X)$ and density-dependent death rate $A(X)$ are given by

$$B(X) = k_2 - (k_2 - k_1)k\frac{X}{K}, \qquad A(X) = k_1 + (k_2 - k_1)(1 - k)\frac{X}{K},$$

Where K is the population carrying capacity, k_2& k_1 is the intrinsic per-capita birth rate and death rate and k , where $0 \leq k \leq 1$, is a parameter describing the extent that density dependence is expressed in births or deaths. Show that this still gives rise to the standard logistic differential equation

$$\frac{dX}{dt} = rX\left(1 - \frac{X}{K}\right). \quad (7.5)$$

(c) Consider the potentially disastrous effects of overfishing causing a population to become extinct, some governments impose quotas that vary depending on estimates of the population at the current time. One harvesting model that takes this into account is

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) - hN.$$

Then show that the only non-zero equilibrium population is

$$N_e = K\left(1 - \frac{h}{r}\right)$$

At what critical harvesting rate can extinction occur? (7.5)

3. (a) Describe an input-output diagram for an **endemic model** of a disease in a city or country over a long time-scale, where there is no reinfection. Write the differential equations governing the model. (7.5)

(b) State any three preliminary assumptions on which a Predator-Prey model is built. How are the differential equations of Predator-Prey model modified to include the effect of DDT? Write the modified equations and define the new parameters p_1 and p_2. (7.5)

(c) In a particular epidemic model, where the infected individuals eventually recover, the population dynamics are governed by the following system of differential equations:

$$\frac{dS}{dt} = -\frac{\beta SI}{N}, \qquad \frac{dI}{dt} = \frac{\beta SI}{N} - \gamma I$$

Given the parameter values $\beta = 2$ and $\gamma = 0.4$ and assuming that the total population N is 1000, in which initially there is only one infected individual but there are 999 susceptible individuals within the population, discuss the following:

(i). When will the disease become an epidemic. Justify your answer.

(ii). What is the peak number of infected individuals at any given time during the epidemic? (7.5)

4. (a) For the predator-prey model with density dependence for the prey and DDT acting on both species,

$$\frac{dX}{dt} = \beta_1 X\left(1 - \frac{X}{K}\right) - c_1XY - p_1X,$$

$$\frac{dY}{dt} = c_2XY - \alpha_2Y - p_2Y$$

Show that (X, Y) is one equilibrium point, where $X = \frac{\alpha_2+p_2}{c_2}$ and $Y = \frac{\beta_1 X\left(1-\frac{X_e}{K}\right)-p_1}{c_1}$. (7.5)

(b) Suppose two species, X and Y are introduced into a new habitat where they compete for the same food source. The system is governed by:

$$\frac{dX}{dt} = 0.8X - 0.02XY$$

$$\frac{dY}{dt} = 0.5Y - 0.01XY$$

(i). Identify the per-capita birth rates (β_1 *and* β_2) for both species and determine which species would grow faster in the complete absence of the other.

(ii). If the initial populations are $X_0 = 50$ and $Y_0 = 30$, calculate the initial rate of change for species X i.e $\left(\frac{dX}{dt}\right)$. Is the population currently increasing or decreasing? (7.5)

(c) Let R(t) denote the number of soldiers of the red army and B(t) the number of soldiers of the blue army. Assuming one army uses aimed fire and the other uses random fire, describe the mathematical model with equations. How does the model change if both armies use random fire? (7.5)

5. (a) The growth of a virus culture is given below:

t (hours)	1	3	5	7	9
P	5	20	55	110	300

Assume the model $P = ae^{bt}$. Transform the data using natural logarithms. Plot the transformed data on a semi-log graph. Estimate the values of a and b. (7.5)

(b) The following data is given for variables x and y:

x	1	2	3	4
y	3	10	21	40

Formulate the least-squares quadratic model of the form: $y = c_1x^2 + c_2x + c_3$ that best fits the data. (7.5)

(c) The following data is given:

x	1	2	3	4	5
y	2	5	10	17	26

Fit the data using the least-squares method for $y = ax + b$. (7.5)

6. (a) What is random number? Use the middle-square method to generate 10 random numbers using $x_0 = 1007$. (7.5)

(b) Using Monte Carlo simulation, write an algorithm to estimate the volume of the region inside the sphere $x^2 + y^2 + z^2 \leq 4$ that lies in the first octant $x > 0, y > 0, z > 0$. (7.5)

(c) Use Monte Carlo simulation to approximate the probability of heads occurring when n times a fair coin is tossed. (7.5)