[This question paper contains 2 printed pages.]

Your Roll No...............

Sr. No. of Question Paper : 5416 **L**

Unique Paper Code : 2354000016

Name of the Paper : Rings and Fields

Name of the Course : **Common Prog Group (Generic Elective)**

Semester : VIII

Duration : 3 Hours Maximum Marks : 90

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. **All** questions are compulsory and carry equal marks.
3. Attempt **two** parts from each question.

1. **(a)** Let R be the ring of all real-valued continuous functions defined on the closed unit interval. Let $M = \{ f \in R \mid f(y) = 0 \}$, where $y \in [0, 1]$. Show that M is a maximal ideal of R.

(b) Let R be a Euclidean ring, and let $a, b \in R$. If $b \neq 0$ is not a unit in R, then $d(a) < d(ab)$.

(c) Show that the ideal generated by a prime element in a commutative ring with unity is a prime ideal.

2. **(a)** Let D be a principal ideal domain and let $p \in D$. Prove that $\langle p \rangle$ is a maximal ideal in D if and only if p is irreducible.

(b) In a Euclidean ring, prove that any two greatest common divisors of a and b are associates.

(c) Let R be a Euclidean ring. Then any two elements $a, b \in R$ have a greatest common divisor d. Moreover, $d = \lambda a + \mu b$ for some $\lambda, \mu \in R$.

3. **(a)** If p is a prime number of the form $4n+1$, then prove that $p=a^2+b^2$ for some integers a and b.

(b) Let $F[x]$ be the field of polynomial rings and if $f(x)$ and $g(x)$ are two nonzero elements of $F[x]$, then $deg(f(x)g(x))=deg(f(x))+deg(g(x))$.

(c) Prove that x^2+x+4 is irreducible over F, the field of integers mod 11 and show that $\frac{F[x]}{(x^2+x+4)}$ is a field having 121 elements.

4. (a) (I) Let p is a prime number, then prove that the polynomial x^n-p is irreducible over the rationals.

(II) The polynomial x^2+4 is irreducible or reducible over the field of rationals. Justify your answer.

(b) (I) Prove that a principal ideal ring is a unique factorization domain.

(II) Prove that $J[\sqrt{-3}]=\{a+b\sqrt{-3} \mid a, b \in Z\}$ be the ring of Gaussian integers is not a unique factorization domain.

(c) Let F be a field and f(x) a non-constant polynomial in $F[x]$. Then prove that there is an extension field E of F in which $f(x)$ has a zero. Also find the splitting field of x^4+1 over Q.

5. (a) (i) If E is a finite extension of F then E is an algebraic extension of F.

(ii) Find the value of [GF(729) : GF(9)].

(b) (i) Draw a subfield lattice of $GF(3^{18})$ and $GF(2^{30})$.

(ii) Define constructible number.

(c) Let x be a zero of $f(x) = x^2+x+1$ in some extension of Z_2. Find the other zero of f(x) in $Z_2[x]$.

6. (a) If a and b are constructible numbers give a geometric proof that a+b and a−b are constructible.

(b) For each prime p and each positive integer n there is, up to isomorphism, a unique finite field of order p^n.

(c) Show that $Q[\sqrt{2}, \sqrt{3}] = Q[\sqrt{2} + \sqrt{3}]$.