[This question paper contains 4 printed pages.]

Your Roll No...............

Sr. No. of Question Paper : 2558 **L**

Unique Paper Code : 2353200011

Name of the Paper : Introduction to Mathematical Modeling

Name of the Course : **B.A. (P) / B.Sc. (Physical Science & Mathematical Science) – DSE**

Semester : VI

Duration : 3 Hours Maximum Marks : 90

<u>Instructions for Candidates</u>

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Attempt **all** question by selecting **two** parts from each question.
3. **All** questions carry equal marks.
4. Use of Scientific Calculator is allowed.

1. (a) The following differential equation describes the level of pollution in the lake

$$\frac{dC}{dt} = \frac{F}{V}C_{in} - C$$

Where V is the volume F is the flow (in and out), C is the concentration of pollution at time t and C_{in} is the concentration of pollution entering lake. Let $V = 28 \times 10^6\ m^3, F = 4 \times 10^6\ m^3$/month. If only freshwater enters the lake.

(i). How long would it take for the lake with pollution concentration 10^7 parts/m^3 to draw below the safety threshold 4×10^6 parts/m^3?

(ii). How long will it take to reduce the pollution level to 5% of its current level? (7.5)

(b) Solve the following potassium-argon decay model

$$\frac{dK(t)}{dt} = -(k_1 + k_2)K,$$

$$\frac{dA(t)}{dt} = k_1K,$$

$$\frac{dC(t)}{dt} = k_2K$$

Where $K(t), A(t)$ and $C(t)$ represent the potassium, argon and calcium in the sample of rock respectively. Then

(i). Solve the system to find $K(t), A(t)$ and $C(t)$ in terms of k_1, k_2 and $k_3 = k_1 + k_2$, using the initial conditions $K(0) = k_0, A(0) = C(0) = 0$.

(ii). Show that $K(t) + A(t) + C(t) = k_0$ for all $t \geq 0$. (7.5)

(c) Write down the word equations which describe the movement of the drug between the two compartments of the body, the GI-Tract and the bloodstream when the patient takes the single pill.

(i). From the word equation, develop the differential equation system which describes this process, defining all variables and parameters as required.

(ii). The constants of proportionality associated with the rate at which the drug diffuses from the GI-Tract into the blood stream, and then is removed from the blood stream are $0.72/\text{hour}$ and $0.15/\text{hour}$ respectively. Suppose initially the amount of the drug in the GI-Tract is 0.0001 mg and there is none in the blood stream. Determine the level of the drug in the blood stream after 6 hours. (7.5)

2. (a) A mouse population, initially consisting of 1,000 mice, has a per-capita birth rate of 8 mice per month (per mouse) and a per-capita death rate of 2 mice per month (per mouse). Also, 20 mouse traps are set each week and they are always filled. Draw a compartment diagram and write a word equation describing the rate of change in the number of mice and then formulate a differential equation for the population with an initial condition. (7.5)

(b) Consider the fish populations as an alternative to the discrete logistic equation. The difference equation is

$$\frac{dN}{dt} = Ne^{r(1-N/K)}, \qquad r > 0,$$

Where N is the population size in generation k, r is the intrinsic growth rate and K is the carrying capacity, which are positive constants. Find all the equilibrium solutions and determine their stability. (7.5)

(c) Consider the discrete logistic equation (with $K = 1$)

$$\frac{dN_n}{dt} = N_n + rN_n(1 - N_n).$$

Show that every second term in the sequence $N_0, N_1, N_2, \ldots$ satisfies the difference equation

$$N_{n+2} = (1 + 2r + r^2)N_n - (2r + 3r^2 + r^3)N_n^2 + (2r^2 + 2r^3)N_n^3 - r^3N_n^4$$

And for equilibrium solution, let $S = N_{n+2} = N_n$ and obtain a quartic equation. (7.5)

3. (a) Formulate the differential equations of a SIR-model. State the assumptions on which it is based. What are its limitations? (7.5)

(b) Find the basic reproduction number r_0 for the SIR model. What is the condition for disease outbreak to become an endemic? If $r_0 = 15$, what proportion of susceptibles need to be vaccinated. (7.5)

(c) Formulate the differential equations for the prey and predator model. Mention the assumptions taken to formulate the problem. (7.5)

4. (a) Consider the following model for two competing species, with densities, X(t) and Y (t), given by the differential equations

$$\frac{dX}{dt} = \beta_1 X - d_1 X^2 - c_1 XY$$

$$\frac{dY}{dt} = \beta_2 Y - d_2 Y^2 - c_2 XY$$

with parameter values $\beta_1 = 3$, $\beta_2 = 3$, $d_1 = 2$, $d_2 = 1$, $c = 2$ and $c_2 = 2.5$. What is the carrying capacity for each of the species, evaluated for the given parameter values? (7.5)

(b) State Gause's Principle of Competitive Exclusion. Extend the competition model to account for logistic growth in both species, in the absence of the other species. (7.5)

(c) Develop a model (a pair of differential equations) for a battle between two armies where both groups use aimed fire. Assume that the red army has a significant loss due to disease, where the associated death rate (from disease) is proportional to the number of soldiers in that army. (7.5)

5. (a) The following data shows the growth of a bacteria culture over time:

t (days)	2	4	6	8	10
P (number of bacteria)	5	15	45	105	175

Assume the model $P = ae^{bt}$. Transform the data using natural logarithms. Plot the transformed data on a semi-log graph. Estimate the values of a and b. (7.5)

(b) For the following dataset:

x	1	2	3	4	5
y	2	5	12	20	40

Assume a linear model $y = ax + b$. Formulate the model that minimizes the maximum deviation between the data points and the line. Estimate the values of a and b. (7.5)

(c) The following data is given:

x	1	2	3	4	5
y	3	12	18	27	42

Fit the data using the least squares method for $y = ax^2$. (7.5)

6. (a) Generate 10 random numbers using the seed $x_0 = 12467$. (7.5)

   (b) Use Monte-Carlo simulation to approximate the area under $f(x) = x^2$ over the interval $0 \leq x \leq 1$. (7.5)

   (c) Find the area enclosed between the curves $y = x^2$ and $y = 4 - x$ within the region bounded by $x = 0$ and $y = 0$. (7.5)