[This question paper contains 4 printed pages.]

Your Roll No................

Sr. No. of Question Paper : 3048 **L**

Unique Paper Code : 2353200011

Name of the Paper : Introduction to Mathematical Modeling

Name of the Course : **B.A. (P) / B.Sc. (Physical Science & Mathematical Science) – DSE**

Semester : VI

Duration : 3 Hours Maximum Marks : 90

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Attempt **all** question by selecting **two** parts from each question.
3. **All** questions carry equal marks.
4. Use of Scientific Calculator is allowed.

1. (a) A certain city had a population of 25,000 in 1960 and a population of 30,000 in 1970. Suppose that its population will continue to grow exponentially at a constant rate. What population can its city planner expect in the year 2000? (7.5)

(b) Consider the following system of equations

$$\frac{dx}{dt} = I - k_1x, \qquad x(0) = 0,$$

$$\frac{dy}{dt} = k_1x - k_2y, \qquad y(0) = 0.$$

Where $x(t)$ and $y(t)$ represent the amount of drug in GI-Track and blood at time t respectively. Solve the above system of equations and write the behaviour of solution as t tends to infinity. (7.5)

(c) Suppose a public bar has a floor area of $20m$ by $15m$ and a height of $4m$, which is continuously with ventilators that exchange the smoke-air mixture with fresh air. Cigarette smoke contains 4% carbon monoxide and a prolonged exposure to a concentration of more than 0.012% can be fatal. If smoke enters the room at constant rate of $0.006m^3/min$, and that the ventilators remove the mixture of smoke and air at 10 times the rate at which smoke is produced. Answer the following

(i). Draw the compartmental diagram and word equation for the changing concentration of carbon monoxide in the bar over time.

(ii). Solve the equation, establish at what time the lethal limit will be reached.

(7.5)

2. (a) Consider the population model

$$\frac{dX}{dt} = rX(X-a)\left(1-\frac{X}{K}\right),$$

Where r is the intrinsic growth rate, K is the carrying capacity and a is a positive constant where $a < K$. Answer the following

(i). Find all the equilibrium populations.

(ii). Determine the stability of each of the three equilibrium points. (7.5)

(b) Solve the following density dependent population growth model:

$$\frac{dX}{dt} = rX\left(1-\frac{X}{K}\right), \quad X(0) = x_0.$$

And

(i). Find the equilibrium points, if any.

(ii). Determine whether the equilibrium points are stable or unstable. (7.5)

(c) Consider the following density dependent population growth model with harvesting

$$\frac{dX}{dt} = rX\left(1-\frac{X}{K}\right) - h, X(0) = x_0,$$

Where $r = 1, K = 10, h = \frac{9}{10}$. Find the equilibrium points and examine whether the population increases or decreases when

(i). $x_0 < 1$
(ii). $1 < x_0 < 9$

(iii). $x_0 > 9$ (7.5)

3. (a) Construct a compartmental diagram for the SIR model and develop appropriate word equations for the rates of change of susceptibles and contagious infectives. Also write the governing equations. (7.5)

(b) Assuming we could instantaneously vaccinate a proportion of a population; what proportion would result in the eradication of the infectious disease? Write the formula using the concept of the basic reproduction number R_0. Further if $R_0 = 4$, what proportion of the population of susceptibles need to be vaccinated? (7.5)

(c) Describe the Lotka–Volterra predator-prey system. Explain the limiting cases of prey with no predators and predators with no prey. (7.5)

4. (a) Suppose two species, X and Y are introduced into a new habitat where they compete for the same food source. The system is governed by:

$$\frac{dX}{dt} = 0.8X - 0.02XY$$

$$\frac{dY}{dt} = 0.5Y - 0.01XY$$

(i). Identify the per-capita birth rates (β_1 *and* β_2) for both species and determine which species would grow faster in the complete absence of the other.

(ii). If the initial populations are $X_0 = 50$ and $Y_0 = 30$, calculate the initial rate of change for species X i.e $\left(\frac{dX}{dt}\right)$. Is the population currently increasing or decreasing? (7.5)

(b) State Gause's Principle of Competitive Exclusion. Extend the competition model to account for logistic growth in both species, in the absence of the other species. (7.5)

(c) Determine the appropriate input-output diagram and associated word equations for the number of soldiers in both the red and blue armies. Formulate the differential equations. What are the assumptions on which it is based? (7.5)

5. (a) The following data represent the growth of a population of algae over a 5-week period:

t (days)	7	14	21	28	35
P (population of algae)	10	55	140	235	290

Test the straight line model $P = c_1 t$ by plotting an appropriate set of data. Estimate the parameter $c_1 t$. (7.5)

(b) The following data is given for the variables x and y:

x	0	1	2	3	4	5
y	3	9	24	45	69	86

Formulate the least squares mathematical model of the form

$$y = c_1x^2 + c_2x + c_3$$

that minimizes the sum of the squared deviations between the data and the model.

(7.5)

(c) The following data is given:

$x\ (\times 10^{-3})$	6	12	18	24	30	36	42	48	54	60
$y\ (\times 10^{-5})$	5	13	28	36	52	78	99	123	140	164

Fit the data using the least squares method for the model $y = ax$. (7.5)

6. (a) Using Monte Carlo simulation, write an algorithm to calculate the volume trapped between the two paraboloids

$$z = 9 - x^2 - y^2 \text{ and } z = x^2 + y^2$$

Note that the two surfaces intersect on the circle $x^2 + y^2 = 3$. (7.5)

(b) Use the linear congruential method to generate 10 random numbers using

$$a = 1, b = 7 \text{ and } c = 8.$$ (7.5)

(c) What is random number? Use the middle-square method to generate 10 random numbers using $x_0 = 3014$. (7.5)