[This question paper contains 4 printed pages.]

Your Roll No................

Sr. No. of Question Paper : 277 **L**

Unique Paper Code : 2353010014

Name of the Paper : Integral Transforms

Name of the Course : **Bachelor of Science (Honours Course) Mathematics**

Semester : VI – DSE

Duration : 3 Hours Maximum Marks : 90

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Attempt any **two** parts from each of the question.

1. (a) Find the Fourier sine series for the function

$$f(x) = x(\pi - x), \qquad 0 < x < \pi$$

and hence deduce that

$$\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots\cdots = \frac{\pi^4}{96}.$$

(b) Obtain the complex Fourier series of

$$f(x) = e^{-|x|}, \qquad -\pi < x < \pi$$

and use Parseval's identity to show that

$$\sum_{k=-\infty}^{\infty} \frac{1}{(1+k^2)^2} = \frac{\pi}{2}\coth \pi.$$

(c) Using the Fourier integral theorem, show that

$$f(x) = \begin{cases} 1 - \dfrac{x}{\pi}, & 0 \le x \le \pi \\ 0, & \text{otherwise} \end{cases}$$

can be represented as

$$f(x) = \frac{1}{\pi}\int_0^{\infty} \frac{1-\cos(a\pi)}{a^2}\cos(ax)\,da.$$

Hence, evaluate

$$\int_0^{\infty} \frac{1-\cos a}{a^2}\,da.$$

2. (a) Show that the Fourier series of the function

$$f(x) = \begin{cases} x, & 0 \le x \le \pi \\ 2\pi - x, & \pi \le x \le 2\pi \end{cases}$$

is

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\cdots\right).$$

Hence, prove that

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.$$

(b) State and prove the pointwise convergence theorem.

(c) Find the Fourier series of

$$f(x) = x^2, \quad -\pi < x < \pi$$

and hence deduce

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

7.5, 7.5, 7.5

3. (a) Define the Fourier transform of a function $f(t)$ and show that

$$\mathcal{F}\{\chi_{[-a,\ a]}(x)\} = \sqrt{\frac{2}{\pi}}\left(\frac{\sin ak}{k}\right),$$

where $\chi_{[-a,\ a]}(x) = H(a - |x|) = \begin{cases} 1, & |x| < a \\ 0. & |x| > a \end{cases}$

(b) Let $f(x)$ and its first derivative vanish as $x \to \infty$. If $F_c(k)$ is the Fourier cosine transform, then show that

$$\mathcal{F}_c[f''(x)] = -k^2 F_c(k) - \sqrt{\frac{2}{\pi}} f'(0).$$

(c) If $F(k)$ and $G(k)$ are Fourier transform of $f(x)$ and $g(x)$ respectively, then show that the Fourier transform of the convolution $(f * g)$ is the product $F(k)G(k)$. 7.5, 7.5, 7.5

4. (a) Show that:

(i) $\mathcal{L}[t^2 e^t] = \frac{2}{(s-1)^2}$

(ii) $\mathcal{L}[e^{at} \cos \omega t] = \frac{s-a}{(s-a)^2+\omega^2}$.

(b) Let f be piecewise continuous in the interval $[0, T]$ for every positive T, and let f be of exponential order, that is, $f(t) = O(e^{at})$ as $t \to \infty$ for some $a > 0$. Then, show that the Laplace transform of $f(t)$ exist for Re $s > a$.

(c) Let $\bar{f}(s)$ be the Laplace transform of $f(t)$ and let f be of exponential order as $t \to \infty$, then show that

$$\mathcal{L}\left[\int_0^t f(\tau)d\tau\right] = \frac{\bar{f}(s)}{s}.$$

7.5, 7.5, 7.5

5. (a) Solve the following ordinary differential equation:

$$-u'' + u = f(x), \qquad -\infty < x < \infty,$$

where $u(x)$ vanishes as $|x| \to \infty$, using Fourier transform method.

(b) Find the solution of the Laplace equation in the half-plane,

$$u_{xx} + u_{yy} = 0, \qquad -\infty < x < \infty, \qquad y \geq 0$$

with the boundary conditions

$$u(x, 0) = f(x), \qquad -\infty < x < \infty,$$

$u(x, y) \to 0$ as $|x| \to \infty$, $y \to \infty$.

(c) Find the solution of the problem of an infinite medium which slows neutrons, in which a source of neutrons is located,

$$u_t = u_{xx} + \delta(x)\,\delta(t), \qquad -\infty < x < \infty, \qquad t > 0,$$

$$u(x, 0) = \delta(x), \qquad -\infty < x < \infty,$$

$$u(x, t) \to 0 \text{ as } |x| \to \infty,$$

where $u(x, t)$ represents the number of neutrons per unit volume per unit time and $\delta(x)\,\delta(t)$ represents the source function. 7.5, 7.5, 7.5

6. (a) Using Laplace transform, solve the third order ordinary differential equation

$$(D^3 + D^2 - 6D)x(t) = 0,$$

$D \equiv \frac{d}{dt}$, $t > 0$, with the initial condition $x(0) = 1$, $x'(0) = 0$ and $x''(0) = 5$.

(b) Use Laplace transform method to solve the first order partial differential equation

$$xu_t + u_x = x, \qquad x > 0, \qquad t > 0,$$

with the initial and boundary conditions $u(x, 0) = 0$ for $x > 0$ and $u(0, t) = 0$ for $t > 0$.

(c) Find the displacement of a semi-infinite string which is initially at rest in its equilibrium position by solving the one-dimensional wave equation

$$u_{tt} = c^2 u_{xx}, \qquad 0 \leq x < \infty, \qquad t > 0,$$

with the initial and boundary conditions $u(0, t) = A\, f(t)$ for $t \geq 0$, $u(x, t) \to 0$ as $x \to \infty$ for $t \geq 0$, and $u(x, 0) = 0 = u_t(x, 0)$ for $0 < x < \infty$, where A is a constant.

7.5, 7.5, 7.5