[This question paper contains 4 printed pages.]

Your Roll No...............

Sr. No. of Question Paper : 1883 **L**

Unique Paper Code : 2353012005

Name of the Paper : Mathematical Modeling

Name of the Course : **B.Sc. (H) Mathematics – DSE**

Semester : IV

Duration : 3 Hours Maximum Marks : 90

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Each question contains **three** parts. Attempt **all** questions by selecting **two** parts from each question.
3. Parts of questions to be attempted together.
4. **All** questions carry equal marks.
5. Use of non-programmable scientific calculators is allowed.

1. (a) (i) A ball and bat together cost \$1.10. The bat cost \$1.00 more than the ball. How much does the ball cost.

(ii) Consider the decay mode $\frac{dc}{dt} = -\lambda c, c > 0$, Introduce suitable scaled variables to nondimensionalize the equation and obtain the dimensionless form of the model.

(b) Using the SIR model,

$$\frac{dS}{dt} = \frac{-\beta SI}{N}, \frac{dI}{dt} = \frac{\beta SI}{N} - \gamma I, \frac{dR}{dt} = \gamma I$$

(i) Discuss the effect of reducing the basic reproduction number R_0 on the epidemic curve. Explain how this leads to flattening of the curve.

(ii) Discuss mathematically how control measures such as social distancing, vaccination, and quarantine lead to flattening of the epidemic curve.

(c) Formulate a Susceptible–Exposed–Infectious–Recovered (SEIR) epidemic model by dividing the total population $N(t)$ into four compartments based on disease status. Assume that individuals enter the susceptible class at a constant recruitment rate, and that all compartments experience a natural death rate. Further, suppose that susceptible individuals become exposed through contact with infectious individuals, and exposed individuals progress to the infectious class before eventually recovering.

(i) Write down the system of first-order differential equations describing the SEIR model under these assumptions.

(ii) Derive the differential equation governing the total population $N(t)$ and obtain its exact solution.

(iii) Construct a discretized version of the SEIR model using a Nonstandard Finite Difference (NSFD) scheme.

2.(a) Construct a dieting model for body weight dynamics by assuming that the body mass of an individual at time t is denoted by $M(t)$. Let $F(t)$ represent the food or nutritional intake, $E(t)$ denote the physical activity level, and let the metabolic rate be described by a function $W(M)$ that depends only on the body 1mass. Using an appropriate balance between intake, expenditure, and metabolism, obtain a differential equation governing the evolution of body mass. Obtain an analytic approximation solution for $M(t)$ by assuming two different ansatz function.

(b) Construct the susceptible–infected–recovered (SIR) epidemic model using the SIR methodology. Obtain a relation between the infected population $I(t)$ and the recovered population $R(t)$. Hence determine the threshold value S^* and discuss the following cases:

(i) $S_o > S^*$: Show that this condition leads to an epidemic.
(ii) $S_o < S^*$: Show that under this condition no epidemic occurs.

(c) Examine an epidemic model in which the susceptible, infected, and recovered population are denoted by $S(t)$, $I(t)$, and $R(t)$, respectively. The transmission process is assumed to depend on the square-root interaction between susceptible and infected individuals, while recovery occurs at a rate proportional to the square root of the infected population. The governing equations of the model are

$$\frac{dS}{dt} = -\beta\sqrt{S}\sqrt{I}, \frac{dI}{dt} = \beta\sqrt{S}\sqrt{I} - \gamma\sqrt{I}, \frac{dR}{dt} = \gamma\sqrt{I},$$

Here, β and γ are positive constants.

3.(a) Consider that the populations $x(t)$ and $y(t)$ satisfies the competition system

$$\frac{dx}{dt} = 14x - 2x^2 - xy$$
$$\frac{dy}{dt} = 16y - 2y^2 - xy$$

(i) Show that the critical points are $(0,0)$, $(0,8)$, $(7,0)$ and $(4,6)$.

(ii) Find the linearization of the above system at each of the critical points.

(iii) Show that the behavior of the point (4,6) is a nodal sink.

(b) Find the critical points of the system

$$\frac{dx}{dt} = -x - y + x^2$$

$$\frac{dy}{dt} = x - y$$

and examine their stability and asymptotic stability using the linearization method.

(c) Show that the following system of differential equations is a case of survival of single Species by investigating the behavior of the critical points (0,0), (0,32), (28,0) and (12,8)

$$\frac{dx}{dt} = 14x - \frac{1}{2}x^2 - xy$$

$$\frac{dy}{dt} = 16y - \frac{1}{2}y^2 - xy$$

4. (a) Write the definitions of stability and asymptotic stability of a critical point of an almost linear system. Considering a mass m that oscillates without damping on a spring with Hooke's constant k, with its position function $x(t)$ satisfies the differential equation $x'' + \omega^2 x = 0$, show that each trajectory other than the critical point (0,0) is an ellipse.

(b) Consider the following system of differential equation

$$\frac{dx}{dt} = -3x + x^2 - xy$$

$$\frac{dy}{dt} = -5y + xy$$

Show that the linearization of the system at (5,2) is $u' = 5u - 5v, v' = 2u$. Find the eigen values corresponding to the coefficient matrix of the linear system and hence confirm the nature of the critical points.

(c) Consider an undamped non-linear spring-mass system. Where the mass m has displacement $x(t)$ from its equilibrium position with the restoring force as $F(x) = -kx + \beta x^3$ for $k \geq 0$.

(i) Find the total energy E and discuss stability for hard and soft spring cases.

(ii) Find all critical points and determine the behavior by converting the system to first-order form for $m = 1, k = 4, B = 1$.

5. (a) Using Monte Carlo Simulation, write an algorithm to calculate an approximation to π by considering the number of random points selected inside the quarter circle

$$Q: x^2 + y^2 = 1, x \geq 0, y \geq 0\,,$$

where the quarter circle is taken to be inside the square $S: 0 \leq x \leq 1$ and $0 \leq y \leq 1$.

(b) Explain the middle-square method of generating random numbers. Give the drawbacks of this method. Find 10 random numbers using $x_0 = 3043$ in the interval (0,1).

(c) Use Monte Carlo Simulation to approximate the probability of three heads occurring when five fair coins are flipped.

6. (a) Find both the maximum solution and minimum solution using graphical analysis of the Linear Programming Problem.

$$\begin{aligned} &\mathit{Optimize}\ Z = 6x + 4y \\ \text{s. to}\quad & -x + y \leq 12, \\ & x + y \leq 24 \\ & 2x + 5y \leq 80 \\ & x, y \geq 0. \end{aligned}$$

Determine the sensitivity of the optimal solution to a change in c_2 using the objective function $6x + c_2y$.

(b) Solve the following Linear Programming Problem using algebraic method and find the optimal solution:

$$\begin{aligned} &\text{Maximize } Z = 3x + 2y \\ \text{s. to}\quad & x + y \leq 4, \\ & x + 3y \leq 6 \\ & x, y \geq 0. \end{aligned}$$

(c) Use the Simplex method to solve the following problem

$$\begin{aligned} &\text{Maximize } Z = 3x + y \\ \text{s. to}\quad & 2x + y \leq 6 \\ & x + 3y \leq 9 \\ & x, y \geq 0. \end{aligned}$$