[This question paper contains 2 printed pages.]

Your Roll No...............

Sr. No. of Question Paper	**:**	**2448**	**L**
Unique Paper Code	:	2352572401	
Name of the Paper	:	Abstract Algebra	
Name of the Course	:	**B.A./B.Sc. (Programme) with Mathematics as Non-Major/ Minor – DSC**	
Semester	:	IV	

Duration : 3 Hours Maximum Marks : 90

<u>Instructions for Candidates</u>

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Attempt **all** questions by selecting **two** parts from each question.
3. **All** questions carry equal marks.
4. The use of a calculator is not allowed.

1. (a) Define an abelian group. Show that the set of all 2×2 matrices with real entries is not a group under multiplication. (7.5)

   (b) Let $G = \left\{\begin{pmatrix} a & a \\ a & a \end{pmatrix} : a \neq 0,\ a \in \mathrm{R}\right\}$. Prove that G is a group under matrix multiplication. (7.5)

   (c) Find the inverse of $\begin{bmatrix} 3 & 4 \\ 4 & 4 \end{bmatrix} \in \mathrm{SL}(2, Z_5)$. (7.5)

2. (a) What is a subgroup? Let G be an abelian group with identity e. Then show that $H = \{x \in G \mid x^2 = e\}$ is a subgroup of G. (7.5)

   (b) Let G be an abelian group and H and K be subgroups of G. Then show that $HK = \{hk \mid h \in H, k \in K\}$ is a subgroup of G. (7.5)

   (c) In the group Z_{12}, find $|a|$, $|b|$ and $|a + b|$ when $a = 5$, $b = 4$. (7.5)

3. (a) Let $\alpha = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 4 & 6 & 2 & 5 \end{bmatrix}$ and $\beta = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 6 & 3 & 5 & 1 & 2 \end{bmatrix}$. (7.5)
   Compute $\alpha^{-1}, \beta^{-1}, O(\alpha), O(\beta)$ and $\beta\alpha$.

   (b) List the elements of the subgroups $\langle 3\rangle$ and $\langle 15\rangle$ in Z_{18}. Let a be a group element of order 18. List the elements of the subgroups $\langle a^3\rangle$, and $\langle a^{15}\rangle$. (7.5)

   (c) State Lagrange's Theorem. Suppose that K is a proper subgroup of H and H is a proper subgroup of G. If $|K| = 42$ and $|G| = 420$, what are the possible orders of H. (7.5)

4. (a) Prove that the group $SL(2, \boldsymbol{R})$ of 2×2 matrices with determinant 1 is a normal subgroup of $GL(2, \boldsymbol{R})$, the group of 2×2 matrices with nonzero determinant. (7.5)

(b) Define factor group. What is the order of the factor group $Z_{60}/\langle 15 \rangle$. (7.5)

(c) Let ϕ be a homomorphism from a group G to a group $\overline{G}$ and let H be a subgroup of G. Then show that if H is cyclic then $\phi(H)$ is also cyclic. Check whether the mapping $\phi(x) = x^2$ from **R**, the real numbers under addition, to itself is a homomorphism? (7.5)

5. (a) (i) Prove that the set $A = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \mid a, b \in \mathbb{Z} \right\}$ is a subring of the ring of all 2×2 matrices over $\mathbb{Z}$. (7.5)

(ii) Prove that a unit of a ring divides every element of the ring.

(b) Prove that the ring $\mathbb{Z}_p$ of integers modulo p where p is a prime is an integral domain. Is $\mathbb{Z} \oplus \mathbb{Z}$ an Integral Domain? Justify. (7.5)

(c) Prove that characteristic of an Integral Domain is *0* or a prime number. (7.5)

6. (a) Let $f: R \to S$ be a ring homomorphism. Prove that (7.5)

(i) $f(A)$ is a subring of S where A is a subring of R.

(ii) if A is an ideal and f is onto S, then f(A) is an ideal of S.

(b) (i) Prove that every ring homomorphism f from Z_n to itself has the form $f(x) = ax$, where $a^2 = a$. (7.5)

(ii) Prove or disprove that the correspondence $x \to 5x$ from Z_5 to Z_{10} is a ring homomorphism.

(c) Let $R[x]$ denote the set of all polynomials with real coefficients and let A denote the subset of all polynomials with constant term *0*. Prove that A is an ideal of $R[x]$ and $A = \langle x \rangle$. (7.5)