[This question paper contains 2 printed pages.]

Your Roll No...............

Sr. No. of Question Paper : 1752 **L**

Unique Paper Code : 2352013602

Name of the Paper : Advanced Linear Algebra

Name of the Course : **B.Sc. (H) Mathematics (NEP-UGCF 2022)**

Semester : VI

Duration : 3 Hours Maximum Marks : 90

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. **All** questions are compulsory.
3. Attempt any **TWO** parts from each question.
4. **All** questions carry equal marks.
5. Use of Calculator is not allowed.

1. (a) Let β and β' be two ordered bases for a finite-dimensional vector space V.
   (i) Define change of coordinate matrix Q that changes β'-coordinates into β-coordinates.
   (ii) Show that for any $v \in V$, $[v]_\beta = Q[v]_{\beta'}$.
   (iii) Let $\beta = \{(1,1),(1,-1)\}$ and $\beta' = \{(2,4),(3,1)\}$ be two ordered bases of $\mathbb{R}^2$. Taking $v = (2,3)$, verify that $[v]_\beta = Q[v]_{\beta'}$.

   (b) Let $T: P_1(\mathbb{R}) \to \mathbb{R}^2$ be a linear transformation given by
   $$T(p(x)) = (p(0), p(2)), \forall\, p(x) \in P_1(\mathbb{R}).$$
   Let β and γ be standard ordered bases for $P_1(\mathbb{R})$ and $\mathbb{R}^2$ respectively. Verify that $[T^t]_{\gamma^*}^{\beta^*} = \left([T]_\beta^\gamma\right)^t$.

   (c) Define annihilator W^0 of a subset W of a vector space V. Show that W^0 is a subspace of V. Prove that if W is a subspace of V, then $dim(W) + dim(W^0) = dim(V)$.

2. (a) Define eigenvalues, eigenvectors and characteristic polynomial of a linear operator T defined on an n-dimensional vector space V. Let T be the operator on $\mathbb{R}^2$ that rotates the plane through an angle θ $(0 < \theta < \pi)$. Obtain the characteristic polynomial of T and hence deduce that T has no eigenvalues and eigenvectors.

   (b) Show that the characteristic polynomial of any diagonalizable linear operator splits. Is the converse true? Justify.

   (c) Let $W_1, W_2, \dots, W_k$ be subspaces of a finite-dimensional vector space V. Show that $V = W_1 \oplus W_2 \oplus \dots \oplus W_k$ if and only if each vector $v \in V$ can be uniquely written as $v = v_1 + v_2 + \dots + v_k$ *for some* $v_i \in W_i$.

P.T.O.

3. (a) For which of the following linear operator T on the vector space V, determine whether the given subspace W is T-invariant subspace of V:
(i) $V = P_2(\mathbb{R})$, $T(f(x)) = f'(x)$, and $W = \{f \in P_2(\mathbb{R}) : f(0) = 1\}$.
(ii) $V = \mathbb{R}^3$, $T(a,b,c) = (a+b, b+c, a+c)$, and $W = \{(t,t,t) : t \in \mathbb{R}\}$.

(b) Let T be a linear operator on a finite dimensional vector space V and let W be a T-invariant subspace of V. Prove that if the characteristic polynomial of T splits, then the characteristic polynomial of T_W also splits.

(c) Compute the minimal polynomial of the matrix A,
$$A = \begin{pmatrix} 3 & -1 & 0 \\ 0 & 2 & 0 \\ 0 & -1 & 2 \end{pmatrix}.$$

4. (a) Let $x = (2, 1+i, -i)$ and $y = (2+i, 2, 1-2i)$ be vectors in C^3. Compute $\langle x, y \rangle$, $\|x\|$, $\|y\|$ and $\|x+y\|$. Then verify the Cauchy-Schwarz inequality and the triangle inequality for above vectors x and y.

(b) Prove that if V is an inner product space, then $|\langle x,y \rangle| = \|x\|\|y\|$ if and only if one of the vectors x or y is a multiple of the other.

(c) Using Gram-Schmidt process, obtain an orthonormal basis β for span(S) where $S = \{(1,0,1), (0,1,1), (1,1,0)\}$. Also, compute the Fourier coefficients of $x = (1,1,1)$ with respect to β.

5. (a) Write the definition of the orthogonal complement of a set S in an inner product space V. Let $W = \text{span}\{(i, 0, 1)\}$ in C^3. Find orthonormal bases for W and $W^{\perp}$.

(b) For each of the following inner product spaces V and linear operators T on V, evaluate T^* at the given vector in V:
(i) $V = \mathbb{R}^2, T(a,b) = (a+b, a-b)$, $x = (1,2)$.
(ii) $V = P_1(\mathbb{R})$ with $\langle f(x), g(x) \rangle = \int_{-1}^{1} f(t)g(t)dt$, $T(f) = f'$, $f(t) = 1+2t$.

(c) Find the minimal solution for the following system of linear equations
$$\begin{aligned} x + 2y + z &= 4 \\ x - y + 2z &= -11 \\ x + 5y \quad &= 19. \end{aligned}$$

6. (a) Let T be a linear operator on a finite dimensional complex inner product space V. Prove that T is normal if and only if there exists an orthonormal basis for V consisting of eigenvectors of T.

(b) For the given matrix A, find an orthogonal matrix P and a diagonal matrix D such that $P^*AP = D$, where
$$A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}.$$

(c) Let T be a linear operator on an inner product space V. Prove that T is an orthogonal projection if and only if T has an adjoint T^* and $T^2 = T^* = T$.