This question paper contains 3 printed pages]

Roll No.

S. No. of Question Paper	:	**8060**
Unique Paper Code	:	**2353010024**
Name of the Paper	:	**DSE 6-(iv) : Geometry of Curves and Surfaces**
Name of the Course	:	**B.Sc. (Honors) Mathematics**
Semester	:	**VIII**

Duration : **3 Hours** Maximum Marks : **90**

(Write your Roll No. on the top immediately on receipt of this question paper.)

Attempt all *six* (6) questions by doing any *two* parts from each question.

All questions carry equal marks.

Notations used are standard.

1. (*a*) Show that the curve $x = a \cosh t$, $y = a \sinh t$ becomes the hyperbola $x^2 - y^2 = 1$ and then determine its curvature.

   (*b*) Prove that curvature of a curve $\boldsymbol{p}(s)$ is identically 0 if and only if $\boldsymbol{p}(s)$ is a line.

   (*c*) Define the terms vertex of a plane curve and rotation index of a closed curve. Give an example of a plane curve $\boldsymbol{p}(s)$ with rotation index 0 and a plane curve $\bar{\boldsymbol{p}}(s)$ with rotation index 4. Is it possible to gradually deform $\boldsymbol{p}(s)$ to $\bar{\boldsymbol{p}}(s)$? Justify your answer.

P.T.O.

2. (a) Assume that the curvature of a space curve $\boldsymbol{p}(s)$ is everywhere positive. Then show that the torsion is zero everywhere if and only if $\boldsymbol{p}(s)$ lies on a plane.

(b) Find the curvature and torsion of the helix

$$x = a \cos t,\ y = a \sin t,\ z = bt \ (a > 0).$$

(c) Define the term total curvature of a closed space curve. Derive the total curvature formula for a space curve $\boldsymbol{q}(t)$ $(a \le t \le b)$, where t is any parameter (not necessarily the length parameter).

3. (a) For each of the following surfaces, write the name of the surface and its equation in cartesian form :

(i) $x = a(u + v),\ y = b(u - v),\ z = 4uv.$

(ii) $x = a\left(\dfrac{u-v}{u+v}\right),\ y = b\left(\dfrac{uv-1}{u+v}\right),\ z = c\left(\dfrac{uv-1}{u+v}\right).$

(b) Prove that a surface is a plane if its second fundamental form **II** vanishes everywhere.

(c) Find the umbilic points of an ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ $(a > b > c > 0)$.

4. (a) For Right helicoid $x = u \cos v,\ y = u \sin v,\ z = f(v)$, determine which part of the sphere is covered by Gauss' spherical map.

(b) Define the exterior differentiation d for 0-form, 1-form and 2-form. Also, prove that

$$dd\theta = 0$$

when θ is of degree 0.

(*c*) Derive the second structure equation of the surface for Exterior Differential Forms

$$dw_2{}^1 = K\theta^1 \wedge \theta^2,\ K = b_{11}b_{22} - b_{12}b_{21}.$$

5. (*a*) Prove that every geodesic of a sphere is a great circle.

(*b*) Determine the connection form $w_2{}^1$ and the curvature K when a Riemannian metric is given by

$$dudu + Gdvdv$$

with respect to the parallel coordinate system.

(*c*) Let C : $u = 0$ be a geodesic in the construction of the geodesic parallel coordinate system and $g = \sqrt{G}$ in the Riemannian metric $dudu + Gdvdv$. Prove that $g_u(0, v) = 0$.

6. (*a*) State Gauss-Bonnet Theorem. Verify it for a sphere with parametric equations.

$$x = a \cos u \cos v, \quad y = a \cos u \sin v, \quad z = a \sin v.$$

(*b*) Prove that :

$$\int_A K\theta^1 \wedge \theta^2 + \int_{\partial A} k_g ds = \sum_{i=1}^{4} l_i - 2\pi$$

when A is a quadrilateral and l_1, l_2, l_3, l_4 are the angles at its vertices.

(*c*) Suppose that A Riemannian metric is defined on a rectangular domain in the (u, v) - plane and its Gaussian curvature is negative or zero. Prove that two geodesics can never intersect at two points such that they together enclose a simply connected domain.