[This question paper contains 3 printed pages.]

Roll No. ______________________

Sr. No. of Question Paper: 8121

Unique Paper Code: 2223010041

Name of the Paper: Group Theory & Applications

Name of the Course: B.Sc. (Hons.) Physics NEP: UGCF-2022

Semester: VIII (DSE Paper)

Duration: 3 Hours **Maximum Marks : 90**

Instructions for Candidates

1. Write your Roll No. on top of this question paper, immediately on its receipt.
2. Attempt FIVE questions in all.
3. Question No. 1 is COMPULSORY.
4. Attempt any FOUR more questions from the rest of the paper.

1. Answer any SIX out of the following seven questions:

(a) Show that the subset $G = \{1, -1, i, -i\}$ of the set of complex numbers is a group under the usual complex multiplication.

(b) If a group contains elements a and b such that order of element a; O(a) = 4, and order of element b; O(b) = 2 and $a^3b = ba$, find O(ab).

(c) Let $G = Z_8$ is a set of integers modulo 8 and $H = \{0, 4\}$. Find all left cosets of H in G.

(d) Prove that if a group G has order 4, then it must be abelian.

(e) State Lagrange's Theorem for a group.

(f) Check whether the set $G = \{1, 2, 3, 4, 5\}$ is a group with respect to multiplication modulo 6 or not.

(g) What is a Lie algebra? State the Jacobi identity.

$(3 \times 6 = 18)$

2.

(a) Let Z denote the set of all integers. Prove that (Z, $*$) forms an abelian group under the binary operation defined by

$$a * b = a + b + 1, \quad \text{for all } a, b \in Z.$$

(6)

(b) Show that H $= \{0, 2, 4\}$ is a subgroup of the group G $= \{0, 1, 2, 3, 4, 5\}$ under addition modulo 6. (6)

(c) Consider the dihedral group $D_4 = \{e, r, r^2, r^3, s, sr, sr^2, sr^3\}$, representing the symmetries of a square. Determine whether D_4 is a cyclic group. (6)

3.

(a) Let G $= S_3$, the symmetric group on three elements:

$$G = \{e, (12), (13), (23), (123), (132)\}.$$

Construct the Cayley table of S_3. Whether S_3 is an abelian group or not. (9)

(b) What is a group isomorphism? Show that the groups $(\{1, -1\}, \times)$ and $(\{0, 1\}, +_2)$ are isomorphic. (9)

4.

(a) Determine whether the following permutations are even or odd:

(i) (135)
(ii) (1356)
(iii) (13567) (6)

(b) Define a representation of a finite group.

Consider a representation of a group G acting on a 3-dimensional space with basis $\{e_1, e_2, e_3\}$. Under a group element R, the basis vectors transform as:

$$Re_1 = e_1, \quad Re_2 = e_2, \quad Re_3 = -e_3$$

(i) Write the matrix representation of R
(ii) Find its character $\chi(R)$.
(iii) Determine whether the representation is reducible or irreducible. (12)

5.

(a) Consider the set $SO(2) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ $\theta \in R$.

Show that SO(2) is a Lie group and find its dimension. (9)

(b) What is SU (2)? Explain its connection with quantum spin. For two spin-$\frac{1}{2}$ particles, determine the possible total spin states and classify them according to their total spin. (9)

6.

(a) Distinguish between point groups, Bravais lattices, and space groups in crystallography. Explain how each of these is used to describe the symmetry of crystals.

(6)

(b) Consider the group $G = \{1, \omega, \omega^2\}$, where ω is a complex cube root of unity satisfying $\omega^3 = 1$ and $\omega \neq 1$.

(i) Determine all conjugacy classes of G.

(ii) Find the number and dimensions of all irreducible representations of G.

(iii) Construct the complete character table of G.

(iv) Verify the orthogonality relations of the characters. (12)