(This Question paper contains 4 printed page) Roll No.............

S. No. Of Question Paper	:	**8119**
Name of the Course	:	**B.Sc. Hons.–(Physics)_NEP: UGCF-2022**
Semester	:	**VIII- Semester**
Name of the Paper	:	**Electrodynamics (DSE Paper)**
Unique Paper Code	:	**2223010039**

Duration : 3 Hours **Maximum Marks: 90**

Instructions

1. Write your Roll Number on the top immediately on the receipt of the question paper.
2. Attempt five questions in all. Question number 1 is compulsory. Attempt any four questions amongst questions 2-6.
3. All questions carry equal marks.
4. Use of non-programmable scientific calculators is allowed.
5. Unless specified, the paper uses SI units.
6. The metric used is $g_{\mu\nu} = (+,-,-,-)$.
7. Students are advised to mention the units and metric they are using.
8. Some useful formulae are provided at the end of question paper.

Q1. Attempt any six parts. All parts carry equal marks.

a) The electric and magnetic fields in an inertial frame have values $\vec{E} = (0, 3, 0)$V/m and $\vec{B} = (0, 0,\ 3 \times 10^{-8})$T respectively. Determine whether there exists a frame in which the magnetic field vanishes.

b) Obtain the condition satisfied by the scalar function χ in order to preserve the Lorenz gauge condition under the gauge transformation $A^\mu \to A^\mu + \partial^\mu \chi$.

c) Show that the homogeneous Maxwell's equation $\partial_\mu \tilde{F}^{\mu\nu} = 0$, $\tilde{F}^{\mu\nu}$ being the dual electromagnetic tensor, can be also written in the form $\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0$.

d) In a velocity selector, the fields are $\vec{E} = 2.5 \times 10^4\, \hat{j}$ V/m, $\vec{B} = 0.08\, \hat{k}$ T. Find the speed of particles that pass undeflected.

e) Show that dipole radiation vanishes if the charge distribution is symmetric under spatial inversion, i.e. $\rho(-\vec{r}) = \rho(\vec{r})$.

f) A non-relativistic charge oscillates as $x(t) = x_0 \cos\omega t$, where x_0 is the amplitude and ω is the angular frequency. Find the instantaneous radiated power $P(t)$ using Larmor's formula.

g) A system is described a Lagrangian density $\mathcal{L}(\phi, \partial_\mu \phi)$. If under an infinitesimal transformation of field $\phi \to \phi + \epsilon\delta\phi$, the Lagrangian density changes by $\delta\mathcal{L} = \epsilon\partial_\mu K^\mu$, show that the Euler-Lagrange equation satisfied the system does not change.

(3 × 6 = 18)

Q2.

a) Suppose that the four-potentials $A^\mu \equiv \left(\frac{\Phi}{c}, \vec{A}\right)$ do not satisfy Lorenz gauge condition. Show that it is always possible to find a scalar field χ such that the gauge-transformed potential $A'^\mu = A^\mu - \partial^\mu \chi$ satisfies the Lorenz condition. Derive the differential equation satisfied by the scalar field χ.

b) Using Maxwell's equations, show that once the Lorenz condition is imposed, Φ and $\vec{A}$ satisfy non-homogeneous wave equations.

c) Evaluate the Lorentz invariant quantity $I = \det(F^{\mu}_{\ \nu})$ in terms of the electric and magnetic fields.

(5, 5, 8)

Q3.

a) A particle of mass m and charge q is moving with velocity $\vec{v}$ in the presence of electric field $\vec{E}$ and magnetic field $\vec{B}$. Write down the covariant form of Lorentz-force equation for this particle and use it to derive the relativistic equation of motion for a charged particle in an electromagnetic field. Show that in the non-relativistic limit, this equation reduces to

$$m\frac{d\vec{v}}{dt} = q(\vec{E} + \vec{v} \times \vec{B})\,.$$

b) A proton gyrates in magnetic field $\vec{B} = B_0(1 + \alpha\, y)\,\hat{k}$, where $B_0 = 0.3$ T and $\alpha = 0.5$ m^{-1}. The component of the proton velocity perpendicular the magnetic field is $v_\perp = 3 \times 10^5$ m/s. Using the guiding-center approximation, calculate (i) the gradient drift velocity, (ii) the drift direction for proton vs. electron, and (iii) the net current density associated with the grad-B drift if the proton and electron number densities are both $n = 10^{16}$ m^{-3}.

(10, 8)

Q4.

a) An electric dipole, placed at the origin and oscillating harmonically along the z-axis is given by $p(t) = p_0 \cos(\omega t)\ \hat{k}$, with $p_0 = q_0 d$. The radiation zone fields of the dipole are

$$\vec{E}_{\text{rad}} = -\frac{\mu_0 p_0 \omega^2}{4\pi}\frac{\sin\theta}{r}\cos\left[\omega\left(t - \frac{r}{c}\right)\right]\hat{\theta},$$

$$\vec{B}_{\text{rad}} = -\frac{\mu_0 p_0 \omega^2}{4\pi c}\frac{\sin\theta}{r}\cos\left[\omega\left(t - \frac{r}{c}\right)\right]\hat{\phi}.$$

Obtain the total time-averaged power radiated by the dipole and describe the angular distribution of the radiation.

b) A point charge q is at rest at the origin of an inertial frame K' moving with constant velocity $\vec{v} = v_0\hat{\imath}$ relative to the lab frame K. In the frame K', the four-potential is given by

$$A'^{\mu} = \left(\frac{\Phi'}{c}, \vec{A}'\right), \quad \text{with} \quad \Phi' = \frac{q}{4\pi\epsilon_0 r'}, \quad \vec{A'} = \vec{0}$$

where $r' = \sqrt{x'^2 + y'^2 + z'^2}$. Using a Lorentz transformation, obtain the four-potential A^{μ} in the lab frame K. Express your answer in terms of the lab coordinates $\vec{r} = (x, y, z, t)$, and hence determine the scalar and vector potentials explicitly.

(10, 8)

Q5.

a) Derive Larmor's formula

$$P = \frac{q^2 a^2}{6\pi\epsilon_0 c^3}$$

for the power P radiated by a non-relativistic accelerating point charge q moving with acceleration a. An electron undergoes simple harmonic oscillation with amplitude 0.1 nm and angular frequency $\omega = 10^{15}$ rad/s. Using Larmor's formula, estimate the total power radiated.

b) Starting from the retarded scalar and vector potentials

$$\Phi(\vec{r}, t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}', t_r)}{|\vec{r} - \vec{r}'|}\, \mathrm{d}^3 r', \text{ and } \quad \vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}', t_r)}{|\vec{r} - \vec{r}'|}\, \mathrm{d}^3 r',$$

derive the Liénard-Wiechert potentials

$$\Phi(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{R(1-\vec{\beta}\cdot\hat{n})}\right]_{\text{ret}}, \qquad \vec{A} = \frac{\mu_0 c}{4\pi}\left[\frac{q\vec{\beta}}{R(1-\vec{\beta}\cdot\hat{n})}\right]_{\text{ret}}$$

for a point charge moving along the trajectory $\vec{r_q}(t)$. Here $[\cdots]_{\text{ret}}$ means the quantity is evaluated at retarded time, $\vec{R} = \vec{r} - \vec{r_q}(t_r)$, $R = |\vec{R}|$, and $\vec{\beta}(t) = \frac{\vec{v}(t)}{c}$

(10, 8)

Q6.

a) Write down the relativistic Lagrangian L for a particle of mass m and charge q moving with velocity $\vec{v}$ in an external electromagnetic field described by the scalar potential $\Phi(\vec{r},t)$ and vector potential $\vec{A}(\vec{r},t)$. Explain each term in the Lagrangian. Determine the canonical momentum $\vec{P}$ of the particle and show that the Hamiltonian $H = \vec{P}\cdot\vec{v} - L$ can be written as

$$H = q\Phi + \sqrt{\left(\vec{P} - q\vec{A}\right)^2 c^2 + m^2c^4}\,.$$

b) Explain the physical significance of the components $\boldsymbol{T^{00}}$, $\boldsymbol{T^{0i}}$, and $\boldsymbol{T^{ij}}$ of the electromagnetic energy-momentum tensor.

(10, 8)

<u>Some Useful Expressions:</u>

1. The Electromagnetic Field Tensor $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$

   $F^{0i} = -\frac{E^i}{c} = -F^{i0}$ and $F^{ij} = -\epsilon^{ijk} B_k$ for metric signature $(+ - - -)$.

   $F^{0i} = \frac{E^i}{c} = -F^{i0}$ and $F^{ij} = \epsilon^{ijk} B_k$ for metric signature $(- + + +)$.

2. Dual field tensor: $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}$

3. Lorentz Transformation of Fields for Boost along x-axis:

$$E'_x = E_x \quad E'_y = \gamma(E_y - v\,B_z) \quad E'_z = \left(E_z + v\,B_y\right)$$
$$B'_x = B_x \quad B'_y = \left(B_y + \frac{v}{c^2}E_z\right) \quad B'_z = \gamma\left(B_z - \frac{v}{c^2}E_y\right)$$

4. Energy Momentum Tensor :

$$T^{\mu\nu} = \frac{1}{\mu_0}\left(-F^{\mu\alpha}F^{\nu}_{\alpha} + \frac{1}{4}g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}\right) \text{ with } g_{\mu\nu} = \text{diag}(+,-,-,-)$$

5. Useful Vector Identities:
   a. $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$
   b. $(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) = (\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{d}) - (\vec{a}\cdot\vec{d})(\vec{b}\cdot\vec{c})$
   c. $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = (\vec{a}\cdot\vec{b}\times\vec{d})\vec{c} - (\vec{a}\cdot\vec{b}\times\vec{c})\vec{d}$
   d. $\vec{\nabla}\cdot(\vec{a}\times\vec{b}) = \vec{b}\cdot(\vec{\nabla}\times\vec{a}) - \vec{a}\cdot(\vec{\nabla}\times\vec{b})$
   e. $\vec{\nabla}\times(\vec{a}\times\vec{b}) = \vec{a}(\vec{\nabla}\cdot\vec{b}) - \vec{b}(\vec{\nabla}\cdot\vec{a}) + (\vec{b}\cdot\vec{\nabla})\vec{a} - (\vec{a}\cdot\vec{\nabla})\vec{b}$
   f. $\vec{\nabla}(\vec{a}\cdot\vec{b}) = (\vec{a}\cdot\vec{\nabla})\vec{b} + (\vec{b}\cdot\vec{\nabla})\vec{a} + \vec{a}\times(\vec{\nabla}\times\vec{b}) + \vec{b}\times(\vec{\nabla}\times\vec{a})$

6. Vector Differential Operators in Cylindrical Coordinates (ρ, ϕ, z),

   a. $\vec{\nabla} f = \frac{\partial f}{\partial \rho}\, \hat{\rho} + \frac{1}{\rho}\frac{\partial f}{\partial \phi}\, \hat{\phi} + \frac{\partial f}{\partial z}\, \hat{z}$ for a scalar field $f(\rho, \phi, z)$

   b. $\nabla^2 f = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$ for a scalar field $f(\rho, \phi, z)$

   c. $\vec{\nabla} \cdot \vec{A} = \frac{1}{\rho}\frac{\partial(\rho A_\rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$ for a vector field $\vec{A} = A_\rho\, \hat{\rho} + A_\phi \hat{\phi} + A_z\, \hat{z}$

   d. $\vec{\nabla} \times \vec{A} = \hat{\rho}\left[\frac{1}{\rho}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right] + \hat{\phi}\left[\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right] + \hat{z}\left[\frac{1}{\rho}\frac{\partial(\rho A_\phi)}{\partial \rho} - \frac{1}{\rho}\frac{\partial A_\rho}{\partial \phi}\right]$

7. Vector Differential Operators in Spherical Polar Coordinates (r, θ, ϕ),

   a. $\vec{\nabla} f = \frac{\partial f}{\partial r}\, \hat{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\, \hat{\theta} + \frac{1}{r \sin\theta}\frac{\partial f}{\partial \phi}\, \hat{\phi}$ for a scalar field $f(r, \theta, \phi)$,

   b. $\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}$

   c. $\vec{\nabla} \cdot \vec{A} = \frac{1}{r^2}\frac{\partial(r^2 A_r)}{\partial r} + \frac{1}{r \sin\theta}\frac{\partial(\sin\theta A_\theta)}{\partial \theta} + \frac{1}{r \sin\theta}\frac{\partial A_\phi}{\partial \phi}$ for a vector field $\vec{A} = A_r\, \hat{r} + A_\theta \hat{\theta} + A_\phi \hat{\phi}$

   d. $\vec{\nabla} \times \vec{A} = \hat{r}\frac{1}{r \sin\theta}\left[\frac{\partial(\sin\theta A_\phi)}{\partial \theta} - \frac{\partial A_\theta}{\partial \phi}\right] + \hat{\theta}\, \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial \phi} - \frac{\partial(r A_\phi)}{\partial r}\right] + \hat{\phi}\frac{1}{r}\left[\frac{\partial(r A_\theta)}{\partial r} - \frac{\partial(A_r)}{\partial \theta}\right]$

8. Useful Physical Constants

Quantity	Value (SI Units)
Electron mass	$m_e = 9.11 \times 10^{-31}$kg
Proton Mass	$m_p = 1.67 \times 10^{-27}$kg
Proton charge	$e = 1.6 \times 10^{-19}$ C
Speed of light	$c = 3 \times 10^{8}$ms^{-1}
Vacuum permittivity	$\epsilon_0 = 8.854 \times 10^{-12}$Fm^{-1}
Vacuum Permeability	$\mu_0 = 1.257 \times 10^{-6}$ Hm^{-1}