[This question paper contains 4 printed pages.]

Roll No........................

Sr. No. of Question Paper: 9551
Unique Paper Code: 2223010033

Name of the Paper: Advanced Quantum Mechanics - I (DSE)
Name of the Course: B.Sc. (Hons.) Physics NEP: UGCF
Semester: VIII

Duration: 3 Hours **Maximum Marks:** 90

Instructions for Candidates

1. Write your Roll No. on top of the question paper, immediately on its receipt.
2. Attempt FIVE questions in all. All questions carry equal marks.
3. Question No. 1 is compulsory.
4. Attempt FOUR more questions from the rest of the paper.
5. A few useful formulae are given at the end.

Q.1 Attempt any SIX of the following seven questions: **$3 \times 6 = 18$ marks**

a. Prove that all eigenvalues of a unitary operator lie on the unit circle on the complex plane, i.e. if λ is the eigenvalue of a unitary operator, then $|\lambda|^2 = 1$.

b. Consider a state $|\Psi\rangle = \frac{1}{\sqrt{2}}|\Phi_1\rangle + \frac{1}{\sqrt{5}}|\Phi_2\rangle + \frac{1}{\sqrt{10}}|\Phi_3\rangle$ which is given in terms of three orthonormal eigenstates $|\Phi_1\rangle$, $|\Phi_2\rangle$ and $|\Phi_3\rangle$ of an operator $\hat{B}$ such that $\hat{B}|\Phi_n\rangle = n^2|\Phi_n\rangle$. Find the expectation value of $\hat{B}$ for the state $|\Psi\rangle$.

c. Consider a system whose state is given in terms of an orthonormal set of three vectors: $|\Phi_1\rangle$, $|\Phi_2\rangle$, $|\Phi_3\rangle$ as

$$|\Psi\rangle = \frac{\sqrt{3}}{3}|\Phi_1\rangle + \frac{2}{3}|\Phi_2\rangle + \frac{\sqrt{2}}{3}|\Phi_3\rangle$$

Let an ensemble of 810 identical systems, each one of them in the state $|\Psi\rangle$. If measurements are done on all of them, how many systems will be found in each of the states: $|\Phi_1\rangle$, $|\Phi_2\rangle$ and $|\Phi_3\rangle$?

d. For a Hamiltonian of a system defined as $\hat{H} = \frac{\hat{p}_x^2}{2m} + \hat{V}(x)$, calculate $[\hat{x}, \hat{H}]$. Also, calculate the uncertainty relation between $\hat{x}$ and $\hat{H}$.

e. Prove that the commutation relation of two Pauli operators is

$$[\sigma_i, \sigma_j] = 2\iota\epsilon_{ijk}\sigma_k$$

f. For the case of angular momentum $j = 1$, find the matrices representing the operators $\hat{J}^2$ and $\hat{J}_z$.

g. Let ρ be a density operator. Show that $Tr(\rho^2) \leq 1$, with equality if and only if ρ is a pure state.

Q 2.

a. Consider a system whose Hamiltonian is given by $\hat{H} = \alpha(|\Phi_1\rangle\langle\Phi_2| + |\Phi_2\rangle\langle\Phi_1|)$, where α is a real number having the dimensions of energy and $|\Phi_1\rangle$, $|\Phi_2\rangle$ are normalized eigenstates of a Hermitian operator $\hat{A}$ that has no degenerate eigenvalues.

i. Prove that $\alpha^{-2}\hat{H}^2$ is a projection operator and $\hat{H}$ is not. **4 marks**

ii. Show that $|\Phi_1\rangle$ and $|\Phi_2\rangle$ are not eigenstates of $\hat{H}$. **3 marks**

iii. Calculate the commutators $[\hat{H}, |\Phi_1\rangle\langle\Phi_1|]$ and $[\hat{H}, |\Phi_2\rangle\langle\Phi_2|]$. Then, find the relation that may exist between them. **3 marks**

iv. Find the normalized eigenstates of $\hat{H}$ and their corresponding energy eigenvalues.

4 marks

b. If $|\Phi_1\rangle$, $|\Phi_2\rangle$ and $|\Phi_3\rangle$ are orthonormal, show that the states $|\Psi\rangle = \iota|\Phi_1\rangle + 3\iota|\Phi_2\rangle - |\Phi_3\rangle$ and $|\chi\rangle = |\Phi_1\rangle - \iota|\Phi_2\rangle + 5\iota|\Phi_3\rangle$ satisfy the Schwarz inequality. **4 marks**

Q 3.

a. Consider a system whose initial state $|\Psi(0)\rangle$ and Hamiltonian $\hat{H}$ are given by

$$|\Psi(0)\rangle = \frac{1}{5}\begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}, \qquad \hat{H} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 5 \\ 0 & 5 & 0 \end{pmatrix}$$

i. If a measurement of energy is carried out, what values would we obtain and with what probabilities? **3 marks**

ii. Find the state of the system at a later time t, $|\Psi(\mathrm{t})\rangle$. **3 marks**

iii. Find the total energy of the system at time $t = 0$ and at any later time t.

3 marks

b.

i. Write down the Hamiltonian for a particle of mass m in a one-dimensional harmonic oscillator potential in terms of momentum $\hat{p}_x$ and position $\hat{x}$ operators. **1 mark**

ii. If one defines new operators

$$\hat{b} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \iota\frac{\hat{p}_x}{m\omega}\right), \qquad \hat{b}^{\dagger} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \iota\frac{\hat{p}_x}{m\omega}\right)$$

Show that the Hamiltonian can be expressed as

$$\hat{H} = \frac{\hbar\omega}{2}\left(\hat{b}\hat{b}^{\dagger} + \hat{b}^{\dagger}\hat{b}\right)$$ **3 marks**

iii. Derive the commutation relation $\left[\hat{b}, \hat{b}^{\dagger}\right]$. **3 marks**

iv. Using your result from (iii), show that the Hamiltonian is

$$\hat{H} = \hbar\omega\left(\hat{b}\hat{b}^{\dagger} + \frac{1}{2}\right)$$ **2 marks**

Q 4.

a. If $\hat{A}$ and $\hat{B}$ are Hermitian operators, show that $(\hat{A}\hat{B} + \hat{B}\hat{A})$ is Hermitian and $(\hat{A}\hat{B} - \hat{B}\hat{A})$ is not. **4 marks**

b. Consider a particle subjected to a one-dimensional harmonic oscillator potential. Suppose at $t = 0$ the state vector is given by $e^{-\iota pa/\hbar}|0\rangle$, where p is the momentum operator and a is some number with dimension length. Using the Heisenberg picture, evaluate the expectation value $\langle x\rangle$ for $t \geq 0$. **6 marks**

c. A spin $\frac{1}{2}$ system is known to be an eigenstate of $\boldsymbol{S}\cdot\boldsymbol{n}$ with eigenvalues $\frac{\hbar}{2}$, where $\boldsymbol{n}$ is a unit vector lying in the xz-plane that makes an angle γ with the positive z-axis.

i: Suppose S_x is measured. What is the probability of getting $\frac{\hbar}{2}$? **5 marks**

ii. Evaluate the dispersion in S_x, $\Delta S_x = \sqrt{\langle\hat{S}_x^2\rangle - \langle\hat{S}_x\rangle^2}$. **3 marks**

Q 5.

a. Find the Clebsch-Gordan coefficients associated with the coupling of the spins of the electron and the proton of a hydrogen atom in its ground state. **12 marks**

b. Show that $\Delta J_x \Delta J_y = \frac{\hbar^2}{2}[j(j+1) - m^2]$.

Where, $\Delta J_x = \sqrt{\langle \hat{J}_x^2 \rangle - \langle \hat{J}_x \rangle^2}$ and $\Delta J_y = \sqrt{\langle \hat{J}_y^2 \rangle - \langle \hat{J}_y \rangle^2}$ **6 marks**

Q 6.

a. Consider a system of three noninteracting identical spin $\frac{1}{2}$ particles that are in the same spin state $\left|\frac{1}{2}, \frac{1}{2}\right\rangle$ and confined to move in a one-dimensional infinite potential well of length a: $V(x) = 0$ for $0 < x < a$ and $V(x) = \infty$ for other values of x. Determine the energy and wavefunction of the ground state, the first excited state, and the second excited state. **9 marks**

b. Consider a bipartite two-level $\left(\text{spin } \frac{1}{2}\right)$ system in the state

$$|\Psi\rangle = \frac{1}{\sqrt{3}}(|\uparrow\rangle_A |\uparrow\rangle_B + |\uparrow\rangle_A |\downarrow\rangle_B + |\downarrow\rangle_A |\uparrow\rangle_B)$$

i. Construct the density operator ρ_{AB}. **3 marks**

ii. Obtain the reduced density operator ρ_A. **3 marks**

iii. Determine whether ρ_A represents a pure or mixed state. **3 marks**

Important Formulae

The three Pauli matrices are:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -\iota \\ \iota & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$