(This Question paper contains 4 printed page) Roll No.________

S. No. Of Question Paper	:	**7441**
Name of the Course	:	**B.Sc. Hons.–(Physics)_NEP: UGCF-2022**
Semester	:	**VIII- Semester**
Name of the Paper	:	**Electrodynamics (DSE Paper)**
Unique Paper Code	:	**2223010039**
Question Paper Set No.	:	**Set – A**

Duration : 3 Hours **Maximum Marks: 90**

Instructions

1. Write your Roll Number on the top immediately on the receipt of the question paper.
2. Attempt five questions in all. Question number 1 is compulsory. Attempt any four questions amongst questions 2-6.
3. All questions carry equal marks.
4. Use of non-programmable scientific calculators is allowed.
5. Unless specified, the paper uses SI units.
6. The metric used is $g_{\mu\nu} = (+,-,-,-)$.
7. Students are advised to mention the units and metric they are using.
8. Some useful formulae are provided at the end of question paper.

Q1. Attempt any six parts. All parts carry equal marks.

a) The electric and magnetic fields in an inertial frame have values $\vec{E} = (6,8,0)$V/m and $\vec{B} = (0,0,2\times 10^{-8})$T respectively. Determine whether there exists a frame in which the electric field vanishes.

b) Show that the Lorenz Gauge condition $\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial\Phi}{\partial t} = 0$ can be written in a manifestly covariant form.

c) Express the quantity $\partial_\mu(A_\nu F^{\mu\nu})$ in terms of electric and magnetic fields. Here $F^{\mu\nu}$ is the electromagnetic field tensor satisfying Maxwell's equations in vacuum and A_ν is the corresponding four-potential.

d) A proton moves in a region where the fields are $\vec{E} = 3.0\times 10^4\,\hat{\imath}$ V/m, $\vec{B} = 0.10\,\hat{k}$ T. If the initial velocity is zero determine the drift velocity.

e) In which direction(s) is the radiation intensity maximum and zero for an oscillating electric dipole?

f) What are the conditions that characterise the radiation (far) zone? Mention two approximations which are valid in this zone?

g) A system is described by a Lagrangian $L(q_i, \dot{q}_i, t)$. If under an infinitesimal transformation of generalised coordinates given by $q_i \to q_i + \epsilon\delta q_i$ the Lagrangian changes by $\delta L = \epsilon\frac{dK}{dt}$, show that the Euler-Lagrange equation satisfied the system does not change.

(3 × 6 = 18)

Q2.

a) Suppose that the vector potential $\vec{A}$ satisfies Coulomb gauge condition. Let the current density $\vec{J}$ be decomposed as $\vec{J} = \vec{J}_\ell + \vec{J}_t$ with $\vec{\nabla}\times\vec{J}_\ell = 0$ and $\vec{\nabla}\cdot\vec{J}_t = 0$. Using Maxwell's equations and continuity equation, show that

i. $\frac{1}{c^2}\vec{\nabla}\frac{\partial\Phi}{\partial t} = \mu_0\vec{J}_\ell$

ii. $\frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} - \vec{\nabla}^2 \vec{A} = \mu_0 \vec{J}_t$

b) Starting from the transformation law of electromagnetic field tensor $F^{\mu\nu}$ as a rank-2 tensor under a Lorentz boost along the x-axis, derive the transformation equations for the components of electric field $\vec{E}$ and magnetic field $\vec{B}$.

(10, 8)

Q3.

a) A charged particle moves in uniform, static crossed fields $\vec{E} = E_0\,\hat{j}$ and $\vec{B} = B_0\,\hat{k}$. Determine the trajectory of the particle and derive the $\vec{E} \times \vec{B}$ drift velocity.

b) A proton starts from rest and is accelerated by a uniform electric field $\vec{E} = 10^6$ V/m. Using the relativistic equation of motion, determine its position and velocity at time $t = 10$ ns. Compare your result with the corresponding non-relativistic motion.

(10, 8)

Q4.

a) The electric and magnetic field vectors in an inertial frame K are given by

$$\vec{E} = (0, E_0, 0), \quad \vec{B} = \left(0, \frac{E_0}{c}, 0\right)$$

where E_0 is a constant. Determine the velocity of an inertial frame K', moving along x-axis relative to frame K, such that the transformed fields $\vec{E'}$ and $\vec{B'}$ make an angle 60° with each other.

b) The total power radiated by an accelerated relativistic charge q, moving with velocity $\vec{v}$ and acceleration $\vec{a}$, is given by the Liénard formula

$$P = \frac{1}{4\pi\epsilon_0}\frac{2q^2\gamma^6}{3c}\left[\left|\dot{\vec{\beta}}\right|^2 - \left|\vec{\beta} \times \dot{\vec{\beta}}\right|^2\right], \quad \text{where } \vec{\beta} = \frac{\vec{v}}{c}, \quad \dot{\vec{\beta}} = \frac{\vec{a}}{c}, \quad \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

Obtain the corresponding expressions and interpret the result physically in the following limiting cases:

i. Non-relativistic motion of the charge
ii. Velocity and acceleration of the particle are collinear
iii. Velocity and acceleration of the particle are perpendicular to each other.

(9, 9)

Q5.

a) The scalar potential $\Phi(\vec{r}, t)$ in Lorenz gauge satisfies the inhomogeneous wave equation

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\Phi(\vec{r}, t) = -\frac{\rho(\vec{r}, t)}{\epsilon_0}.$$

For which the Green's function is defined by

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)G(\vec{r}, \vec{r}'; t, t') = -4\pi\delta^3(\vec{r} - \vec{r}')\delta(t - t').$$

i. Using Fourier transform in time and the result

$$(\nabla_{\mathbf{R}}^2 + k^2)\frac{e^{ikR}}{R} = -4\pi\delta^3(\vec{R}), \qquad \vec{R} = \vec{r} - \vec{r}', \qquad R = |\vec{r} - \vec{r}'|$$

(where $\nabla_{\mathbf{R}}^2$ is the Laplacian with respect to $\vec{R}$), show that

$$G_{\text{ret}}(\vec{r}, \vec{r}'; t, t') = \frac{\delta\left(t - t' - \frac{|\vec{r} - \vec{r}'|}{c}\right)}{|\vec{r} - \vec{r}'|}$$

is the retarded Green's function for the wave equation.

ii. Why is the retarded Green's function physically preferred over the advanced Green's function?

iii. Using $G_{\text{ret}}(\vec{r},\vec{r}\,';t,t')$, derive the expression for the retarded scalar potential:

$$\Phi_{\text{ret}}(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}\,',t_r)}{|\vec{r}-\vec{r}\,'|}\, d^3r', \qquad t_r = t - \frac{|\vec{r}-\vec{r}\,'|}{c}$$

b) An oscillating charge distribution has a characteristic size of 1 cm and radiates at 10 GHz. Estimate the wavelength of the emitted radiation. Hence determine the ranges of the observation distance r from the source corresponding to the near, intermediate, and far (radiation) zones.

(12, 6)

Q6.

a) Write the Lagrangian density for the electromagnetic field interacting with a four-current J^μ. Derive the Maxwell's inhomogeneous equations using the Euler–Lagrange equations for the fields.

b) Show that the trace of the electromagnetic energy-momentum tensor vanishes: $T^\mu_\mu = 0$. What is the physical significance of this result?

(10, 8)

Some Useful Expressions

1. The Electromagnetic Field Tensor $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$

   $F^{0i} = -\frac{E^i}{c} = -F^{i0}$ and $F^{ij} = -\epsilon^{ijk} B_k$ for metric signature $(+\,-\,-\,-)$.

   $F^{0i} = \frac{E^i}{c} = -F^{i0}$ and $F^{ij} = \epsilon^{ijk} B_k$ for metric signature $(-\,+\,+\,+)$!

2. Dual field tensor: $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}$

3. Lorentz Transformation of Fields for Boost along x-axis:

$$E'_x = E_x \quad E'_y = \gamma(E_y - v\,B_z) \quad E'_z = (E_z + v\,B_y)$$
$$B'_x = B_x \quad B'_y = \left(B_y + \frac{v}{c^2}E_z\right) \quad B'_z = \gamma\left(B_z - \frac{v}{c^2}E_y\right)$$

4. Energy Momentum Tensor :

$$T^{\mu\nu} = \frac{1}{\mu_0}\left(-F^{\mu\alpha}F^{\nu}_{\alpha} + \frac{1}{4}g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}\right) \text{ with } g_{\mu\nu} = \text{diag}(+,-,-,-)$$

5. Useful Vector Identities:
   a. $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$
   b. $(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) = (\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{d}) - (\vec{a}\cdot\vec{d})(\vec{b}\cdot\vec{c})$
   c. $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = (\vec{a}\cdot\vec{b}\times\vec{d})\vec{c} - (\vec{a}\cdot\vec{b}\times\vec{c})\vec{d}$
   d. $\vec{\nabla}\cdot(\vec{a}\times\vec{b}) = \vec{b}\cdot(\vec{\nabla}\times\vec{a}) - \vec{a}\cdot(\vec{\nabla}\times\vec{b})$
   e. $\vec{\nabla}\times(\vec{a}\times\vec{b}) = \vec{a}(\vec{\nabla}\cdot\vec{b}) - \vec{b}(\vec{\nabla}\cdot\vec{a}) + (\vec{b}\cdot\vec{\nabla})\vec{a} - (\vec{a}\cdot\vec{\nabla})\vec{b}$
   f. $\vec{\nabla}(\vec{a}\cdot\vec{b}) = (\vec{a}\cdot\vec{\nabla})\vec{b} + (\vec{b}\cdot\vec{\nabla})\vec{a} + \vec{a}\times(\vec{\nabla}\times\vec{b}) + \vec{b}\times(\vec{\nabla}\times\vec{a})$

6. Vector Differential Operators in Cylindrical Coordinates (ρ, ϕ, z),

   a. $\vec{\nabla} f = \frac{\partial f}{\partial \rho}\,\hat{\rho} + \frac{1}{\rho}\frac{\partial f}{\partial \phi}\,\hat{\phi} + \frac{\partial f}{\partial z}\,\hat{z}$ for a scalar field $f(\rho, \phi, z)$

   b. $\nabla^2 f = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$ for a scalar field $f(\rho, \phi, z)$

   c. $\vec{\nabla} \cdot \vec{A} = \frac{1}{\rho}\frac{\partial(\rho A_\rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$ for a vector field $\vec{A} = A_\rho\,\hat{\rho} + A_\phi \hat{\phi} + A_z\,\hat{z}$

   d. $\vec{\nabla} \times \vec{A} = \hat{\rho}\left[\frac{1}{\rho}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right] + \hat{\phi}\left[\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right] + \hat{z}\left[\frac{1}{\rho}\frac{\partial(\rho A_\phi)}{\partial \rho} - \frac{1}{\rho}\frac{\partial A_\rho}{\partial \phi}\right]$

7. Vector Differential Operators in Spherical Polar Coordinates (r, θ, ϕ),

   a. $\vec{\nabla} f = \frac{\partial f}{\partial r}\,\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\,\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\,\hat{\phi}$ for a scalar field $f(r, \theta, \phi)$,

   b. $\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}$

   c. $\vec{\nabla} \cdot \vec{A} = \frac{1}{r^2}\frac{\partial(r^2 A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta A_\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}$ for a vector field $\vec{A} = A_r\,\hat{r} + A_\theta \hat{\theta} + A_\phi \hat{\phi}$

   d. $\vec{\nabla} \times \vec{A} = \hat{r}\frac{1}{r\sin\theta}\left[\frac{\partial(\sin\theta A_\phi)}{\partial \theta} - \frac{\partial A_\theta}{\partial \phi}\right] + \hat{\theta}\,\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial \phi} - \frac{\partial(r A_\phi)}{\partial r}\right] + \hat{\phi}\frac{1}{r}\left[\frac{\partial(r A_\theta)}{\partial r} - \frac{\partial(A_r)}{\partial \theta}\right]$

8. Useful Physical Constants

Quantity	Value (SI Units)
Electron mass	$m_e = 9.11\ \times 10^{-31}$kg
Proton Mass	$m_p = 1.67\ \times 10^{-27}$kg
Proton charge	$e = 1.6\ \times 10^{-19}$ C
Speed of light	$c = 3\ \times 10^{8}$ms^{-1}
Vacuum permittivity	$\epsilon_0 = 8.854\ \times 10^{-12}$Fm^{-1}
Vacuum Permeability	$\mu_0 = 1.257\ \times 10^{-6}$ Hm^{-1}