[This question paper contains 2 printed pages.]

Your Roll No................

Sr. No. of Question Paper : 7389 **L**

Unique Paper Code : 2353200021

Name of the Paper : Rings and Fields

Name of the Course : **Bachelor of Arts/ Bachelor of Science (Programme)**

Semester : VIII

Duration : 3 Hours Maximum Marks : 90

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. **All** questions are compulsory.
3. Attempt any **two** parts from each question.
4. **All** questions carry equal marks.

1. (a) Define ideal of a ring. Let I and J be ideals of a ring R. Prove that the product IJ is an ideal of R.

(b) Define ring homomorphism. Show that the image of a ring homomorphism is a subring of the codomain.

(c) Let $\mathbb{Z}$ denote the ring of integers. Define a mapping $\phi: \mathbb{Z} \to \mathbb{Z}$ by $\phi(n) = 5n$. Verify that ϕ is a ring homomorphism and find ker ϕ.

2. (a) Let R be a Euclidean ring and $a, b \in R$. Prove that the greatest common divisor of a and b can be written in the form $ax + by$ for some $x, y \in R$.

(b) Define maximal ideal of a ring. Find all maximal ideals in the ring $\mathbb{Z}_{30}$.

(c) Show that every Euclidean domain is a unique factorization domain.

3. (a) Prove that the set $J[i] = \{a + bi | a, b \in \mathbb{Z}\}$ forms an Euclidean ring under the usual addition and multiplication of complex numbers.

(b) Let $f(x) = 5x^4 + 3x^3 + 1 \; and \; g(x) = 3x^2 + 2x + 1$ be polynomials in $Z_7[x]$. Determine the quotient and remainder on dividing $f(x)$ by $g(x)$ using division algorithm.

(c) Let F be a field. Prove that if $f(x)$ and $g(x)$ are two non-zero elements of $F[x]$ then $deg(f(x).g(x)) = deg(f(x)) + deg(g(x))$.

4. (a) Let F be a field. Prove that if $f(x)$ is a non constant polynomial in $[x]$, then there is an extension field E of F in which $f(x)$ has a zero.

(b) Define splitting field. Find splitting field for $f(x) = x^4 + 1$ over $\mathbb{Q}$.

(c) Let F be a field. Prove that if $f(x)$ is a non constant polynomial in $F[x]$, then prove that there exists a splitting field E for $f(x)$ over F .

5 (a) Let K be a finite extension field of the field E and let E be a finite extension field of the field F. Then prove that K is a finite extension field of F and $[K: F] = [K: E][E: F]$.

(b) Define minimal polynomial and find the minimal polynomial for $\sqrt{-3} + \sqrt{2}$ over the field $\mathbb{Q}$.

(c) (i) Define an algebraic element and a transcendental element with example of each.

(ii) Is $\mathbb{Q}(\sqrt{2})$ is an algebraic extension of $\mathbb{Q}$? Justify.

6 (a) Show that as a group under multiplication, the set of non-zero elements of the field $GF(p^n) \cong \mathbb{Z}_{p^n-1}$. Hence find out $[GF(729): GF(9)]$.

(b) Write all the elements of $GF(8)$ in additive form corresponding to their multiplicative form.

(c) Use the fact that $4\cos^2\left(\frac{2\pi}{5}\right) + 2\cos\left(\frac{2\pi}{5}\right) - 1 = 0$ to prove that a regular pentagon is constructible.