[This question paper contains 4 printed pages.]

Your Roll No................

Sr. No. of Question Paper : 7383 **L**

Unique Paper Code : 2353010025

Name of the Paper : Integral Equations and Calculus of Variations

Name of the Course : **Bachelor of Science (Honours Course) Mathematics**

Semester : VIII

Duration : 3 Hours Maximum Marks : 90

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. **All** questions carry equal marks.
3. Attempt any **two** parts from each question.

1. (a) Show that

$$y(x) = \cos x$$

satisfies

$$\phi(x) = 1 - \int_0^x (x-t)\phi(t)\,dt.$$

(b) Convert the initial value problem

$$y''(x) = y(x),\ y(0) = 1,\ y'(0) = 0$$

into an integral equation.

(c) Solve the integral equation by successive approximations:

$$\phi(x) = x - \int_0^x (x-t)\phi(t)\,dt, \qquad \phi_0(x) = 0.$$

2. (a) Solve using Laplace transform :

$$\phi(x) = e^{2x} + \int_0^x e^{t-x}\phi(t)\,dt.$$

(b) Solve the integro-differential equation:

$$\phi''(x) - 2\phi'(x) + \phi(x) + 2\int_0^x \cos(x-t)\,\phi''(t)\,dt + 2\int_0^x \sin(x-t)\,\phi'(t)\,dt = \cos(x)$$

$$\phi'(0) = \phi(0) = 0.$$

(c) Solve the generalized Abel equation :

$$f(x) = \int_0^x (x-t)^{\frac{1}{2}}\phi(t)\,dt.$$

3. (a) Show that the function $\varphi(x) = \sqrt{x}$ is a solution of the Fredholm-type integral equation

$$\varphi(x) - \int_0^1 K(x,t)\varphi(t)dt = \sqrt{x} + \frac{x}{15}\left(4x^{\frac{3}{2}} - 7\right)$$

where the kernel is of the form

$$K(x,t) = \begin{cases} \dfrac{x(2-t)}{2}, & 0 \le x \le t \\ \dfrac{t(2-x)}{2}, & t \le x \le 1 \end{cases}$$

(b) Using Fredholm determinates, find the resolvent kernel of the kernel

$K(x,t) = 2x - 1;\ \ 0 \le x \le 1, \qquad 0 \le t \le 1.$

(c) Determine the iterated kernels $K_1(x, t)$ and $K_2(x, t)$ if $K(x, t) = e^{\min(x,t)}$, $a = 0$, $b = 1$.

4. (a) Solve the integral equation

$$\varphi(x) - \lambda\int_0^1 K(x,t)\varphi(t)dt = e^x$$

where

$$K(x,t) = \begin{cases} \dfrac{\sinh(x)\sinh(t-1)}{\sinh(1)}, & \text{if } 0 \le x \le t \\ \dfrac{\sinh(t)\sinh(x-1)}{\sinh(1)}, & \text{if } t \le x \le 1 \end{cases}$$

(b) Define the degenerate kernel of the Fredholm integral equation. Then, solve the integral equation

$$\varphi(x) - \lambda \int_0^{2\pi} (\sin(x)\cos(t) - \sin(2x)\cos(2t) + \cos(3t)\sin(3x))\,\varphi(t)dt = \cos(x)$$

(c) Define the characteristic number and eigenfunction for the Fredholm integral equation. Determine the characteristic numbers and eigenfunction of the following integral equation

$$\varphi(x) - \lambda \int_0^{\pi} (\cos^2(x)\cos(2t) + \cos(3x)\cos^3(t))\,\varphi(t)dt = 0.$$

5. (a) Prove that the necessary condition for the functional

$$J[y] = \int_{x_1}^{x_2} f(x, y, y')\,dx,$$

to be an extremum is that

$$\frac{d}{dx}\left\{f - y'\frac{\partial f}{\partial y'}\right\} - \frac{\partial f}{\partial x} = 0.$$

(b) On which curve the given functional be extremized

$$J[y] = \int_0^{\frac{\pi}{2}} (y'^2 - y^2 + 2xy)\,dx,\ y(0) = 0, y\left(\frac{\pi}{2}\right) = 0.$$

(c) Find the shortest distance between the point P(–1,4) and the straight line y = 1 – 4x.

6. (a) Find the extremals of the functional

$$J[y] = \int_0^{\frac{\pi}{4}} (y''^2 - y^2 + x^2)dx, y(0) = 0, y'(0) = 1, y\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} = y'\left(\frac{\pi}{4}\right).$$

(b) Find the geodesic on a sphere of radius 'a' are its great circles.

(c) Prove that the sphere is the solid figure of revolution, which for a given surface area has maximum volume.