[This question paper contains 2 printed pages.]

Your Roll No................

Sr. No. of Question Paper : 7382 **L**

Unique Paper Code : 2353010024

Name of the Paper : DSE 6 (iv) Geometry of Curves and Surfaces

Name of the Course : **B.Sc. (Honors) Mathematics**

Semester : VIII

Duration : 3 Hours Maximum Marks : 90

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Attempt all **six (06)** questions by doing any **two** parts from each question.
3. **All** questions carry equal marks.
4. Notations used are standard.

Q1 (a) Write the parametric equation of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ traced in the anticlockwise direction and then determine its curvature.

(b) Prove that if the curvature of a plane curve $\boldsymbol{p}$ is constant $\kappa > 0$, then $\boldsymbol{p}$ describes a circle with radius $\frac{1}{\kappa}$.

(c) Prove that every oval (i.e. convex closed curve) has at least four vertices.

Q2 (a) Define the curvature of a space curve $\boldsymbol{p}(s)$ and then state the Frenet-Serret's formula. Also state the assumption required on the curvature, if any.

(b) Calculate the curvature and torsion for the space curve $q(t) = (t, t^2, t^3)$.

(c) If a simple closed curve $\boldsymbol{p}(s)$ in the space is a non-trivial knot, then show that its total curvature is at least 4π.

Q3 (a) For each of the following surfaces, write the name of surface and its equation in cartesian form

(i) $x = au\cos v,\ y = bu\sin v,\ z = u^2$

(ii) $x = a\left(\frac{u-v}{u+v}\right),\ y = b\left(\frac{uv+1}{u+v}\right),\ z = c\left(\frac{uv-1}{u+v}\right)$

(b) Prove that the necessary and sufficient condition for a curve $\vec{p}(s) = \big(u(s), v(s)\big)$ on a surface $\vec{p}(u,v)$ to be a geodesic is that $\big(u(s), v(s)\big)$ is a solution to the following system of differential equations

$$\frac{d^2u}{ds^2}+\Gamma^u_{uu}\frac{du}{ds}\frac{du}{ds}+2\Gamma^u_{uv}\frac{du}{ds}\frac{dv}{ds}+\Gamma^u_{vv}\frac{dv}{ds}\frac{dv}{ds}=0,$$

$$\frac{d^2v}{ds^2}+\Gamma^v_{uu}\frac{du}{ds}\frac{du}{ds}+2\Gamma^v_{uv}\frac{du}{ds}\frac{dv}{ds}+\Gamma^v_{vv}\frac{dv}{ds}\frac{dv}{ds}=0$$

P.T.O.

(c) Prove that the surface

$$x = 3u + 3uv^2 - u^3,\ y = v^3 - 3v - 3u^2 v,\ z = 3\left(u^2 - v^2\right),$$

is a minimal surface. Also, calculate its Gaussian curvature K.

Q4 (a) What is Ruled Surface? Prove that for a ruled surface, $K \le 0$ everywhere.

(b) Define the exterior multiplication. Prove that when the coordinates (u, v) are changed to the polar coordinates (r, θ) by using the equations $u = r\cos\theta,\ v = r\sin\theta$, then

$$du \wedge dv = r\,dr \wedge d\theta.$$

(c) Let $\vec{p}(s)$ be a space curve with curvature $k > 0$ everywhere. Show that the torsion $\tau = 0$ everywhere if and only $\vec{p}(s)$ lies in a plane.

Q5 (a) Let the differential operators X and Y acting on functions be defined as

$$X = \xi^1 \frac{\partial}{\partial u^1} + \xi^2 \frac{\partial}{\partial u^2};\quad Y = \eta^1 \frac{\partial}{\partial u^1} + \eta^2 \frac{\partial}{\partial u^2}.$$

Express the vector field $[X, Y]$ using ξ^i, η^i $(i = 1, 2)$. Prove that for three vector fields X, Y, Z, the following equations holds:

$$[[X,Y],[Z]] + [[Y,Z],[X]] + [[Z,X],[Y]] = 0$$

(b) For a differential 2-form $\varphi = f\,du^1 \wedge du^2$, let $\varphi(x, y) = f\left(\xi^1\eta^2 - \xi^2\eta^1\right)$. Show that if φ and ψ are 1-forms, then

(i) $(\varphi \wedge \psi)(X, Y) = \varphi(X)\psi(Y) - \psi(X)\varphi(Y)$.

(ii) $(d\varphi)(X, Y) = X(\varphi(Y)) - Y(\varphi(X)) - \varphi([X, Y])$.

(c) Define the covariant derivative $\frac{dX}{dt}$ of a tangent vector $X = \xi^1 e_1 + \xi^2 e_2$ along the curve $\vec{p}(t) = \vec{p}(u(t), v(t))$ lying on the surface $\vec{p}(u, v)$. Show that it is independent of the second fundamental form.

Q6 (a) Prove that every geodesic of a sphere is a great circle.

(b) Determine the connection form w_2^1 and the curvature K when a Riemannian metric is given by

$$dudu + Gdvdv$$

with respect to the parallel coordinate system.

(c) Let $C : u = 0$ be a geodesic in the construction of the geodesic parallel coordinate system and $g = \sqrt{G}$ in the Riemannian metric $dudu + Gdvdv$. Prove that $g_u(0, v) = 0$.