Serial No. 9550

[This question paper contains 4 printed pages.]

Roll No.

Sr. No. of Question Paper:	
Unique Paper Code:	2223010033
Name of the Paper:	Advanced Quantum Mechanics - I (DSE)
Name of the Course:	B.Sc. (Hons.) Physics NEP: UGCF
Semester:	VIII

Duration: 3 Hours　　　　Maximum Marks: 90

Instructions for Candidates

1. Write your Roll No. on top of the question paper, immediately on its receipt.
2. Attempt FIVE questions in all.
3. Question No. 1 is compulsory.
4. Attempt FOUR more questions from the rest of the paper.

1. Attempt any SIX of the following seven questions:

(a) Check the validity of the Schwarz inequality in the context of Hilbert space vectors using the ket

$$|1\rangle = \begin{pmatrix} 3 \\ i \end{pmatrix}, |2\rangle = \begin{pmatrix} 2 \\ -4i \end{pmatrix}$$

(b) Let $\hat{D} = d/dx$ be the derivative operator. Find the value of $(e^{\hat{D}})x^3$.

(c) Consider a one-dimensional particle confined within the region $0 \leq x \leq a$ and having wavefunction $\psi(x,t) = \sin(\pi x/a)e^{-i\omega t}$. Find the associated potential $\hat{V}(x)$.

(d) Starting from $[\hat{L}_x, \hat{L}_y]$, obtain the commutation relation between the raising and lowering angular momentum operator, i.e $[\hat{L}_+, \hat{L}_-]$.

(e) What conditions must the parameter α and operator $\hat{G}$ satisfy so that the operator $\hat{U} = e^{i\alpha\hat{G}}$ is unitary?

(f) Write down the normalization condition for discrete basis and continuous basis in quantum mechanics and give an example for each.

(g) Determine whether the density matrix

$$\hat{\rho} = \begin{pmatrix} 0.8 & 0 \\ 0 & 0.2 \end{pmatrix}$$

represents a pure state or mixed state.

$3 \times 6 = 18$ Marks

2.

(a) Consider the Hermitian operator

$$\hat{A} = \begin{pmatrix} 3 & 4 \\ 4 & 3 \end{pmatrix}$$

acting on a two - dimensional Hilbert space. Find the eigenvalues (a_1, a_2) and the corresponding normalized eigenvectors $(|\psi_1\rangle, |\psi_1\rangle)$ of $\hat{A}$. Construct the projection operators $\hat{P}_1$ and $\hat{P}_2$ corresponding to both eigenvectors and verify that $\hat{A} = a_1\hat{P}_1 + a_2\hat{P}_2$.

(12 marks)

(b) Consider four operators $\hat{A}, \hat{B}, \hat{C}$, and $\hat{D}$, where $\hat{A}$ and $\hat{B}$ are Hermitian operators and $\hat{C}$ and $\hat{D}$ are skew-Hermitian operators. If $\hat{Q}_1 = [\hat{B}, \hat{C}]$ and $\hat{Q}_2 = [(\hat{A} + \hat{B}), \hat{D}]$. Identify with proof whether $\hat{Q}_1$ and $\hat{Q}_2$ are Hermitian, skew-Hermitian, or neither.

(6 marks)

3.

(a) The creation and annihilation operators for the one-dimensional harmonic oscillator in the Schrödinger picture are defined as

$$\hat{a}\_ = \frac{1}{\sqrt{2m\omega\hbar}}(m\omega\hat{x} + i\hat{p}), \qquad \hat{a}_+ = \frac{1}{\sqrt{2m\omega\hbar}}(m\omega\hat{x} - i\hat{p})$$

where $\hat{x}$ and $\hat{p}$ satisfy the canonical commutation relation $[\hat{x}, \hat{p}] = i\hbar$.

i) Write down the Hamiltonian of the harmonic oscillator H in Schrodinger picture.
ii) Write down the Heisenberg equation of motion in differential form for the ladder operators $\hat{a}_\pm^H(t)$.
iii) Solve this differential equation to find the explicit time dependence of ladder operators in the Heisenberg picture.

(12 marks)

(b) Show that $\{\hat{\sigma}_i, \hat{\sigma}_j\} = 2\delta_{ij}\hat{I}$, where $\{\hat{\sigma}_i, \hat{\sigma}_j\} = \hat{\sigma}_i\hat{\sigma}_j + \hat{\sigma}_j\hat{\sigma}_i$.

(6 Marks)

4.

(a) Consider the addition of two angular momenta with quantum numbers $j_1 = 1$ and $j_2 = 1$. Write down the possible values of the total angular momentum quantum number j and the allowed values of the magnetic quantum number m for each case. Evaluate the Clebsch-

Gordon coefficients for the coupled states having total angular momentum quantum number j = 2.

(15 marks)

(b) Given the canonical commutation relation $[\hat{x}, \hat{p}] = i\hbar$, evaluate the commutator

$$[\hat{x}, e^{a\hat{p}}]$$

where a is a real constant.

(3 marks)

5.

(a) Two identical, non-interacting spin-0 bosons of mass m are confined in a one-dimensional infinite potential well of width L. Write the normalized two-particle ground-state wavefunction. Find the ground-state energy. Calculate the probability that both particles are found in the right half of the box.

(10 marks)

(b) Using the definitions of angular momentum ladder operators, obtain an expression for the operator product $\hat{L}_+\hat{L}_-$ in terms of $\hat{L}^2$ and $\hat{L}_z$ operators. Also, evaluate the eigenvalue of the operator $\hat{L}_+\hat{L}_-$ operated on the state $|2,1\rangle$.

(8 marks)

6.

(a) Consider a spin-$\frac{1}{2}$ particle described using the standard basis $\{|\uparrow\rangle, |\downarrow\rangle\}$. The system is prepared such that with probability 0.6, the particle is in the normalised state $|\psi_1\rangle$, and with probability 0.4 , it is in the state $|\psi_2\rangle$, where

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle), \qquad |\psi_2\rangle = |\uparrow\rangle$$

Write each state using the column vector representation and construct the corresponding density matrix for the system. Find the eigenvalues of the density matrix. Determine if it represents a pure state or mixed state.

(12 marks)

(b) Assume a spin-$\frac{1}{2}$ particle in the eigenstate of $\hat{S}_z$ with eigenvalue +1 given by the state $|+z\rangle$. Using the generalized uncertainty principle, find the product $\Delta\hat{S}_x\Delta\hat{S}_y$ for the state $|+z\rangle$.

(6 Marks)

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Important formulae

The Pauli matrices are:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$