[This question paper contains 4 printed pages.]

Your Roll No.....

Sr. No. of Question Paper : 10029 **L**

Unique Paper Code : 2354000010

Name of the Paper : Introduction to Mathematical Modeling (GE)(NEP-UGCF)

Name of the Course : **Common Prog Group**

Semester : VI

Duration : 3 Hours Maximum Marks : 90

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. All six questions are compulsory.
3. Attempt any two parts from each question.
4. All questions carry equal marks.
5. Use of non-programmable Scientific Calculator is allowed.

Q1. (a) Consider the lake pollution model governed by the differential equation:

$$\frac{dC}{dt} = \frac{F}{V}c_{in} - \frac{F}{V}C$$

where C(t) is the concentration of pollutant in the lake at time t, F is the flow rate, V is the volume of the lake, and c_{in} is the concentration of pollutant in the inflowing water.

(i) Solve the above differential equation subject to the initial condition $C(0) = c_0$.

(ii) Lake Erie has volume
$V = 458 \times 10^9 m^3$ and flow rate $F = 1.75 \times 10^{11}\ m^3$/year

Using above model, estimate the time (in years) required for the pollution level to reach 5% of its current level if only fresh water flows into the lake.

(b) The Earth's atmospheric pressure P is often modelled by assuming that $\frac{dP}{dh}$ (the rate at which pressure P changes with altitude h above sea level) is proportional to P. Suppose that the pressure at sea level is 1013 millibars and that the pressure at an altitude of 20 km is 50 millibars.

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(i) Solve the differential equation $\frac{dP}{dh} = -kP$. Determine k and the constant of integration from the initial conditions.

(ii) What is the atmospheric pressure at an altitude of 50 km?

(iii) At what altitude is the pressure equal to 900 millibars

(c) Write down the word equations which describe the movement of the drug between the two compartments in the body, the GI-tract and the bloodstream, when a patient takes a single pill.

(i) From the word equations, develop the differential equation system which describes this process, defining all variables and parameters as required.

(ii) The constants of proportionality associated with the rates at which the drug diffuses from the GI-tract into the bloodstream and then is removed from the bloodstream are $0.72\ hour^{-1}$ and $0.15\ hour^{-1}$ respectively. Suppose initially the amount of drug in GI-tract is 0.0001 mg and none in the bloodstream. Determine the level of drug in the bloodstream after 6 hours.

Q2. (a) A population, initially consisting of 1,000 mice, has a per-capita birth rate of 8 mice per month (per mouse) and a per-capita death rate of 2 mice per month (per mouse). Also, 20 mouse traps are set each week and they are always filled. First, write a word equation describing the rate of change in the number of mice and hence formulate and solve the differential equation for the population with an initial condition.

(b) A bacterial culture initially contains x_0 bacteria. After 2 hours, the population doubles.
(i) Write and solve the differential equation describing the growth model.
(ii) Determine the growth constant k.
(iii) Find the population after 6 hours.

(c) Suppose that the population of a country is modelled using the differential equation $\frac{dX}{dt} = 0.2\,X - 0.001\,X^2$, with initial population $X(0) = 100$. The unit of time is in months. Find the population of the country after 2 months.

Q3. (a) Develop a model with three differential equations describing a predator-prey interaction, where there are two different species of prey and one species of predator. Clearly write down the assumptions while developing the model.

(b) Develop a model for a battle between red army and blue army where both groups used aimed fire. Assume that the red army has a significant loss due to disease, where the associated death rate from disease is proportional to the number of soldiers in that army.

(c) Construct a compartmental diagram for the model and develop appropriate word equations for the rates of change in susceptible, contagious infectives and recovered population.

Q4. (a) For the predator-prey model with density dependence for the prey and DDT acting on both species

$$\frac{dX}{dt} = \beta_1 X\left(1 - \frac{X}{K}\right) - c_1 XY - p_1 X, \quad \frac{dY}{dt} = c_2 XY - \alpha_2 Y - p_2 Y,$$

Show that (X, Y) is one equilibrium point, where $X = \frac{\alpha_2 + p_2}{c_2}$, $Y = \frac{\beta_1\left(1 - \frac{X_e}{K}\right) - p_1}{c_1}$ and determine whether there are any other equilibrium points.

(b) Determine the appropriate input-output diagram and associated differential equation for the number of soldiers in both the red and blue armies using aimed fire, and modify the model to account for both of the armies using random fire.

(c) Determine the predator and prey model in the limiting cases of prey with no predator and predator with no prey.

Q5. (a) Given the following data:

x	2	3	4
y	5	7	9

Formulate the mathematical model that minimizes the sum of absolute deviations between the observed data and the model $y = cx$. Set up the objective function and residuals, but do not solve for c.

(b) Use the middle-square method to generate 10 random numbers using $x_0 = 3043$.

(c) Using Monte Carlo simulation, write an algorithm to estimate the volume of the part of the sphere $x^2 + y^2 + z^2 = 1$ that lies in the first octant.

Q6. (a) Fit the straight line $y = ax + b$ to the following data using the method of least squares:

x	1	2	3	4	5
y	1	1	2	2	4

(b) On flipping a die, the probability of each event occurrence is 1/6. Using Monte Carlo Simulation, write Monte Carlo Roll of a Fair Die Algorithm.

P.T.O.

(c) Write an algorithm using Monte Carlo simulation to compute the volume of the part of the ellipsoid,

$$\frac{x^2}{2}+\frac{y^2}{4}+\frac{z^2}{8}\leq 16$$

that lies in the first octant.

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