[This question paper contains 4 printed pages.]

Your Roll No...............

Sr. No. of Question Paper : 10706 **L**

Unique Paper Code : 2353200011

Name of the Paper : Introduction to Mathematical Modeling

Name of the Course : **BA (P) / BSc. (Physical Science & Mathematical Science) – DSE**

Semester : VI

Duration : 3 Hours Maximum Marks : 90

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Attempt **all** question by selecting **two** parts from each question.
3. **All** questions carry equal marks.
4. Use of Scientific Calculator is allowed.

1. (a) An arrow is shot straight upward from the ground with an initial velocity of 160 ft/s. It experiences both the deceleration of gravity and deceleration $v^2/800$ due to air resistance. How hight in the air does it go? (7.5)

(b) Consider the following system of equations

$$\frac{dx}{dt} = -k_3 x, \qquad x(0) = k_0$$

$$\frac{dy}{dt} = k_1 y, \qquad y(0) = 0$$

$$\frac{dz}{dt} = k_2 z, \qquad z(0) = 0$$

Where $x(t), y(t)$ and $z(t)$ represent the potassium, argon and calcium in the sample of rock respectively. Solve the above system of equations and show that the $x(t) \to 0$

$, y(t) \to \frac{k_1 k_0}{k_3}$ and $z(t) \to \frac{k_2 k_0}{k_3}$ as $t \to \infty$. (7.5)

(c) Suppose L (t) denotes the amount of salt in a tank with capacity 1000 litters. An incoming water at a rate 100 litter has the concentration of 10. Assuming that the inflow and outflow rate is same. Write the balance law that captures the rate of change of amount of salt in the tank and derive the differential equation. Further solve the differential equation. (7.5)

2. (a) Consider a population whose initial value is x_0, with constant per-capita birth rate and death rate as k_1 and k_2 respectively, then

(i). Find the population size at any time t.

(ii). Solve the population of the world for continuous population growth.

(iii). Find an expression for the time for the population to double in size. (7.5)

(b) A difference equation describing female insects with a periodic breeding time is

$$\frac{dX}{dt} = rX\left(1 - \frac{X}{K}\right),$$

Where all the parent insects die after laying their eggs.

(i). Where is this model different from the discrete logistic equation in lectures?

(ii). Find the all-equilibrium solution, for $r > 1$, and determine their stability. (7.5)

(c) Let $X(t)$ denotes the population at time t. By taking suitable parameters derive the Logistic Equation with harvesting that captures the density dependent growth rate. (7.5)

3. (a) Write an Input-output diagram for the epidemic model of influenza in a school, where there is no reinfection. Describe the assumptions of the model and the differential equations governing the model. (7.5)

(b) (i). Define the basic reproduction number R_0 in words and provide its mathematical formula in terms of $\beta, \gamma, and\ S_0$, where S_0 is the initial susceptible population.

(ii). A specific strain of measles has an $R_0 = 12.5$. Calculate the minimum proportion P of the population that must be vaccinated to achieve herd immunity and preven an outbreak. (7.5)

(c) In 1932, the Russian microbiologist Gause described an experiment with two strains of yeast, Saccharomyces cervevisiae and Schizosaccharomyces kefir, hereafter called Species X and Species Y. Gause found that, grown on their own, each species exhibited a logistic growth curve, but when grown together, the growth pattern changed with Species X dying out. Formulate the competition model to account for the logistic growth in both species, in the absence of the other species. (7.5)

4. (a) For the predator-prey model with density dependence for the prey and DDT acting on both species,

$$\frac{dX}{dt} = \beta_1 X\left(1 - \frac{X}{K}\right) - c_1 XY - p_1 X,$$

$$\frac{dY}{dt} = c_2 XY - \alpha_2 Y - p_2 Y$$

Find an equilibrium point of the above system. (7.5)

(b) Consider the following model for two competing species, with densities, X(t) and Y(t), given by the differential equations

$$\frac{dX}{dt} = \beta_1 X - d_1 X^2 - c_1 XY$$

$$\frac{dY}{dt} = \beta_2 Y - d_2 Y^2 - c_2 XY$$

with parameter values $\beta_1 = 3$, $\beta_2 = 3$, $d_1 = 2$, $d_2 = 1$, $c = 2$ and $c_2 = 2.5$. What is the carrying capacity for each of the species, evaluated for the given parameter values?

(7.5)

(c) Develop a model (a pair of differential equations) for a battle between two armies where both groups use aimed fire. Assume that the red army has a significant loss due to disease, where the associated death rate (from disease) is proportional to the number of soldiers in that army. (7.5)

5. (a) For the following data set:

x	1	2	3	4	5
y	2	7	15	23	37

Fit the data using the least squares method for the model $y = ax^2$. (7.5)

(b) For the following data set:

x	0.5	1.0	2.4	3.2	4.1	5.0
y	1.6	2.7	3.4	3.9	4.3	4.9

Formulate the mathematical model that minimizes the largest deviation between the data and the line $y = ax + b$. Solve for the estimates of a and b. (7.5)

(c) For the following data:

x	1	2	3	4	5
y	2	7	11	25	50

Test the exponential model $y = ae^{bx}$ by plotting the transformed data. Estimate the parameters a and b. (7.5)

6. (a) Use the middle-square method to generate 15 random numbers using $x_0 = 5072$. (7.5)

(b) Use Monte Carlo simulation to approximate the area under the curve $f(x) = x$ over the interval $0 \leq x \leq 2$. (7.5)

(c) Use the linear congruential method to generate 10 random numbers using

$$a = 5, b = 3, m = 16, x_0 = 2.$$ (7.5)

(500)