[This question paper contains 2 printed pages.]

Your Roll No...............

Sr. No. of Question Paper : 1733 **L**

Unique Paper Code : 2352013601

Name of the Paper : Advanced Group Theory

Name of the Course : **B.Sc. Maths (Hons)**

Semester : VI

Duration : 3 Hours Maximum Marks : 90

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. **All** questions are compulsory & carry equal marks.
3. Attempt any **five** parts from Question **1**. Each part of Question **1** is of **3** marks.
4. Attempt any **two** parts from each of the questions **2** to **6**. Each part is of **7.5** marks.
5. **All** parts of a question are to be attempted together.
6. Use of calculator is not allowed.

1. (a) Let a group G act on a set A. Show that the stabilizer of an element $a \in A$ is a subgroup of G.

   (b) Let a group G act on itself by left multiplication. Find the kernel of the action.

   (c) Prove that if A and B are subsets of G with $A \subseteq B$, then $C_G(B)$ is a subgroup of $C_G(A)$ where $C_G(A)$ denotes the centralizer of a subset A in G.

   (d) Show that the symmetric group S_3 is solvable.

   (e) Let $\rho = (1\ 2\ 3\ 4\ 5) \in S_5$. Find an element $\tau \in S_5$ such that $\tau\rho\tau^{-1} = \rho^2$

   (f) Prove that the product of two characteristic subgroups is a characteristic subgroup.

   (g) Define a nilpotent group. Prove that every abelian group is nilpotent.

2. (a) Prove that actions of a group G on a set A and the homomorphisms from G into the symmetric group S_A are in bijective correspondence.

(b) Prove Lagrange's Theorem for finite groups using Group Actions.

(c) Let G = {1, a, b, c} be the Klein 4-group. Label the vertices 1, a, b, c by 1, 2, 3, 4 respectively. Under this labelling and the action of left multiplication by the group element, give the permutation representation associated to this action.

3. (a) Let G be a group, let H be a subgroup of G and let G act by left multiplication on the set A of left cosets of H in G. Let Π_H be the associated permutation representation afforded by this action. Then prove that

(i) G acts transitively on A

(ii) the stabilizer in G of the point $1H \in A$ is the subgroup H

(iii) the kernel of this action is $\cap_{x \in G} xHx^{-1}$ and $\mathrm{Ker}(\Pi_H)$ is the largest normal subgroup of G contained in H.

(b) Find all conjugacy classes of D_8, the dihedral group of order 8.

(c) Show that $|C_{S_n}(12)(34)| = 8(n-4)!\ \forall\ n \leq 4$. Determine the elements in this centralizer explicitly.

4. (a) Use Sylow's theorem to determine if a group of order 105 is simple or not.

(b) Show that a group of order 56 is not simple.

(c) Prove that if G is a group of order 231 then Z(G), the center of G, contains a Sylow 11-subgroup of G and a Sylow 7-subgroup is normal in G.

5. (a) Let G be a group of order pq where p and q are primes, p < q and p does not divide (q – 1). Then prove that G is a cyclic group. Hence deduce that a group of order 33 is cyclic.

(b) Prove that there is no simple group of order 280.

(c) State and prove Index Theorem. Prove that a group of order 72 is not simple.

6. (a) Let p be a prime. Prove that if G is a non-abelian group of order p^3, then the commutator subgroup G′ is equal to the center Z(G).

(b) Let p and q be primes. Show that any group of order p^2q is solvable.

(c) Let G be a finite nilpotent group. Prove that no proper subgroup H of G is equal to its normalizer N(H).