[This question paper contains 4 printed pages.]

Your Roll No...............

Sr. No. of Question Paper : 10017 **L**

Unique Paper Code : 2354000010

Name of the Paper : Introduction to Mathematical Modeling (GE)(NEP-UGCF)

Name of the Course : **Common Prog Group**

Semester : VI

Duration : 3 Hours Maximum Marks : 90

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. All **six** questions are compulsory.
3. Attempt any **two** parts from each question.
4. **All** questions carry equal marks.
5. Use of non-programmable Scientific Calculator is allowed.

Q1. (a) A population of a certain country grows at a rate of 5% per year. If the initial population is 100, in how much time will the population double?

(b) Suppose a lake has constant volume V and is continuously well mixed. Let C(t) denote the concentration of pollutant in the lake at time t. Let F be the flow rate of water into and out of the lake, and let c_{in} be the concentration of pollutant in the incoming water.

(i) Applying the balance law to the mass of the pollutant M(t), develop the differential equation describing the model.

(ii) Solve the differential equation subject to the initial condition C(0) = c_0

(c) The following model describes the levels of drugs in a patient taking a course of cold pills:

$$\frac{dx}{dt} = I - k_1 x, x(0) = 0$$

$$\frac{dy}{dt} = k_1 x - k_2 y, y(0) = 0$$

where k_1 and k_2 ($k_1 > 0$, $k_2 > 0$ and $k_1 \neq k_2$) describe rates at which the drug moves between the two sequential compartments (the GI-tract and the bloodstream) and I denotes the amount of drug released into the GI-tract in each time step. The levels of the drug in the GI-tract and bloodstream are x and y respectively. Show by solving the equations sequentially that the solutions are:

$$x(t) = \frac{I}{k_1}(1 - e^{-k_1 t})$$

$$y(t) = \frac{I}{k_2}[1 - \frac{1}{k_2 - k_1}(k_2 e^{-k_1 t} - k_1 e^{-k_2 t})]$$

Q2. (a) A population X(t) is modelled by the differential equation

$$\frac{dX}{dt} = rX$$

where r is the growth rate of the population.
(i) Solve the differential equation, given that X(0) = x_0.
(ii) Find an expression for the time required for the population to double in size.
(iii) The world population in 1990 was x_0=5.3 billion and the growth rate is r = 0.017 per year. Using above model, estimate the population after 5 years and 10 years.

(b) Find the equilibrium solutions of the Logistic differential equation and discuss their stability.

(c) Let N(t) be the number of ranchers who have adopted an improved pasture technology in Uruguay. Then N(t) satisfies the differential equation

$$\frac{dN}{dt} = aN(1 - \frac{N}{N_T})$$

where N_T is the total population of ranchers. Taking N_T = 17015, a = 0.490, N_0= 141, determine how long it takes for the improved pasture technology to spread to 80% of the population.

Q3. (a) Construct a compartmental diagram for the model and develop appropriate word equations for the rates of change of susceptible, contagious infectives and recovered populations.

(b) Determine the appropriate input-output diagram and associated differential equation for the number of soldiers in both the red and blue armies using aimed fire, and

(i) modify the model to account for the red army using random fire.

(ii) modify the model to account for reinforcement at a constant rate.

(c) Let $X(t)$ denote the number of preys per unit area and $Y(t)$ the number of predators per unit area.

(i) Make some assumptions to formulate differential equations for prey and predator densities.

(ii) Check the Lotka-Volterra model in the limiting cases of prey with no predators, or predators with no prey.

Q4. (a) Determine a compartment diagram and an appropriate word equation for each of the two populations, the predator and the prey.

(b) (i) Find the equilibrium solutions of the equations $\frac{dX}{dt} = 2X - XY, \frac{dY}{dt} = Y - XY.$

(ii) Find a relation between Y and X by the chain rule for the differential equations $\frac{dX}{dt} = -XY,\ \frac{dY}{dt} = -2Y.$

(c) Consider a disease where all those infected remain contagious for life. Ignore all births and deaths. Write down suitable word equations for the rate of change of the number of susceptible and infectives. Hence, develop a pair of differential equations.

Q5. (a) Using Monte Carlo simulation, approximate the value of π by selecting random points inside the quarter circle $Q: x^2 + y^2 = 1,\ x \geq 0,\ y \geq 0$, where the quarter circle is taken to be inside the square:

$$S: 0 \leq x \leq 1,\ 0 \leq y \leq 1$$

Use the equation $\pi/4 =$ area Q /area S.

(b) Use the middle-square method to generate 10 random numbers using seed $x_0 = 1009$.

(c) For each of the following data sets, formulate the mathematical model that minimizes the largest deviation between the data and the line $y = ax + b$

x	1.0	2.3	3.7	4.2
y	3.6	3.0	3.2	5.1

Q6. (a) Generate ten random numbers using the Linear Congruence Method,

$$x_i \equiv (ax_{i-1} + b)\mathrm{mod}\,(m)$$

where $m = 16$, $a = 5$, and $b = 3$, with seed $x_0 = 3$.

(b) Fit the curve $y = Ax^2$ to the following data using the Least Squares Method:

x	0.5	1.0	1.5	2.0	2.5
y	0.7	3.4	7.2	12.4	20.1

(c) Use Monte Carlo simulation to approximate the probability of heads occurring when a fair coin is tossed n times.