[This question paper contains 2 printed pages.]

Your Roll No...............

Sr. No. of Question Paper : 1714 **L**

Unique Paper Code : 2352014801

Name of the Paper : Field Theory and Galois Extension

Name of the Course : **B.Sc. (Hons.) Mathematics**

Semester : VIII

Duration : 3 Hours Maximum Marks : 90

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. **All** questions are compulsory.
3. Attempt any **two** parts from each question.
4. **All** questions carry equal marks.
5. Use of non- programmable scientific calculator is permitted.
6. All symbols have usual meaning as per prescribed textbook.

1. (a) If M : L and L : K are extensions. Then show that

$$[M : K] = [M : L] [L : K]$$

(b) If $f \in K[x]$,then there exists a splitting field extension $L: K$ for f, with$[L: K] \leq n!$

(c) If L: K is an extension of field K then [L:K]<∞ if and only if there exist finitely many algebraic elements $\alpha_1, \alpha_2, \alpha_3, \ldots\ldots \alpha_n$ in L such that L=K($\alpha_1, \alpha_2, \alpha_3, \ldots\ldots\ldots \alpha_n$).

2. (a) If L: K is an extension of field K and $\alpha \in$ L, then α is algebraic over K if and only if [K(α): K] < ∞. Further deduce that [K(α): K] = deg (m_α) for minimal polynomial m_α.

(b) If $f \in K[x]$ is irreducible of degree n, show that there is a simple extension K(α): K such that [K(α): K] = n and $f(\alpha)$ =0.

(c) (i) Let $f = x^2 + 2x + 3$ be in $Z[x]$, find its splitting field in $\mathbb{C}(x)$. What if $f \in Z_7\ [x]$?

(ii) Can we construct a regular seven -sided polygon with ruler and compasses? Justify.

3. (a) Show that a finite extension $L: K$ is normal if and only if $L: K$ is a splitting field for some $g \in K[x]$.

(b) Define $L: K$ as a separable extension. Show that if $L: K$ is separable and M is an intermediate field ($L: M: K$), then $L: M$ is separable and $M: K$ is separable.

(c) Show that if $L: K$ is an extension and A is a subset of $Aut(L)$, then

(i) $\gamma\phi(A) \supseteq A$

(ii) $\phi\gamma(K) \supseteq K$

(iii) $\phi\gamma\phi(A) = \phi(A)$

(iv) $\gamma\phi\gamma(K) = \gamma(K)$.

4. (a) Suppose $L: K$ is finite. If $L: K$ is a Galois extension, show that $|\gamma(K)| = [L: K]$ and $K = \phi\gamma(K)$. Otherwise, $|\gamma(K)| < [L: K]$ and K is a proper subfield of $\phi\gamma(K)$. State any theorems you use clearly.

(b) Find the Galois group of $x^4 - 2$ over the field $\mathbb{Z}_7$.

(c) Suppose $L: K$ is finite. Let $G = \Gamma(L: K)$ and let $K_0 = \phi(G)$. If $L: M: K$, let $\gamma(M) = \Gamma(L: M)$. Show that ϕ is one-to-one from the set of subgroups of G onto fields M, intermediate between L and K, and γ is its inverse map. State any theorems you use clearly.

5. (a) Define the discriminant of a polynomial. Prove that the discriminant is zero if and only if the polynomial has a repeated root. Compute the discriminant of the polynomial $f(x) = x^3 - 3x + 1$.

(b) Suppose L: K is a splitting field extension for x^m-1 over K. Define m-th-cyclotomic polynomial Φ_m. Prove that $x^m - 1 = \prod_{d/m} \Phi_d$.

(c) Show that the mapping $\Phi(a) = a^p$ is an automorphism of a finite field F of order p^n (p, a prime) and generates Γ(F, Z_p).

6. (a) Show that an algebraic extension L:K is simple if there are only finitely many intermediate fields.

(b) Show that Q($\sqrt{2}, \sqrt{3}$): Q is a simple extension.

(c) Find the Galois groups of x^5+1 over Q.