[This question paper contains 4 printed pages.]

Your Roll No...............

Sr. No. of Question Paper : 1718

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Unique Paper Code : 2372014801

Name of the Paper : Bayesian Inference

Name of the Course : **B.Sc. (Hons.) Statistics under NEP-UGCF**

Semester : VIII

Duration : 3 Hours

Maximum Marks : 90

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Attempt **six** questions in all.
3. Use of a non-programmable scientific calculator is allowed.

1. (a) Describe the role of Bayes' theorem in updating prior beliefs to obtain the posterior distribution of an unknown parameter. Further, explain the advantages of its sequential application.

(b) Suppose $\underset{\sim}{X} = (X_1, X_2, \dots, X_n)$ is a random sample from normal distribution with unknown mean θ and known variance σ^2. Assume that the prior distribution of θ is Normal(μ, τ^2). Obtain the posterior distribution θ given $\underset{\sim}{x}$, and identify the resulting posterior distribution.

(6, 9)

2. (a) Let X be a random variable taking values 0 and 1 with probability mass function

$$f(x|\theta) = \begin{cases} 1-\theta, & x = 0 \\ \theta, & x = 1. \end{cases}$$

Assume that the prior distribution for θ is given by

$$g(\theta) = \begin{cases} 0.25, & \theta = 0.4 \text{ or } 0.6 \\ 0.50, & \theta = 0.5. \end{cases}$$

Obtain the marginal probability mass function of x and the posterior distribution of θ given an observed value of x.

(b) Suppose $X_1, X_2, \ldots, X_n$ is a random sample from Pareto(k, θ), unknown θ. Construct conjugate prior for θ and identify the resulting prior distribution. Also, obtain the posterior distribution of θ given the sample.

(7, 8)

3. (a) Let $X \sim N(\mu, \sigma^2)$. Obtain Jeffreys' non-informative prior for σ in the following cases:

(i) when the mean μ is known, and

(ii) when the mean μ is unknown, assuming that (μ, σ) are a-priori dependent.

(b) Suppose $X_1, X_2, \ldots, X_n$ are independently and identically distributed Bernoulli random variables with an unknown probability of success θ. Obtain the large sample approximation to the posterior distribution of θ, and clearly state the result used in obtaining this approximation. (7, 8)

4. (a) Explain how the Bayes estimate of the unknown parameter θ is obtained under a specified loss function $L(\theta, a)$ using the extensive form of Bayesian analysis. Hence, derive the Bayes estimator of θ under absolute error loss function $L(\theta, a) = |\theta - a|$.

(b) Consider the simple linear regression model $Y_i = \beta X_i + U_i, i = 1,2, \ldots, n$, where Y_i is the dependent variable, X_i is a fixed (non-stochastic) independent variable, and U_i represents the random error term. The regression parameter β is unknown. Assume that the errors U_i are independently and identically distributed as $N(0, \sigma^2)$, with σ^2 known. Further, suppose the prior distribution of β is $N\left(\beta_0, \frac{\sigma^2}{m}\right), m > 0, \beta_0 \in (-\infty, \infty)$. Derive the posterior distribution of β. Show that the posterior mean is a weighted average of the prior mean and the ordinary least squares estimator of β. (6, 9)

5. (a) Assume $X \sim N(\theta, 1)$ and consider the squared error loss function is $L(\theta, a) = (\theta - a)^2$. Let the class of decision rules be given by $\delta_c(x) = cx$.

(i) Show that the decision rule δ_1 is R-better than δ_c for all $c > 1$.

(ii) Show that for $0 \leq c \leq 1$, the decision rules δ_c are non-comparable.

(iii) If the prior distribution of θ is $N(0, \tau^2)$, find the Bayes risk of δ_c. Hence, obtain the Bayes rule and the corresponding Bayes risk under this prior.

(b) Let $X \sim Poisson(\theta)$, where θ is unknown, and suppose the prior distribution for θ is $Gamma(\alpha, \beta)$. Given the loss function

$$L(\theta, \hat{\theta}) = \theta^{a-1} e^{-b\theta} (\theta - \hat{\theta})^2, a > 1, b > 0,$$

find the Bayes estimate of θ using the duality between the prior distribution and the loss function.

(8, 7)

6. (a) Define Bayes factor in favour of H_0. Show that Bayes factor is a ratio of likelihoods. Assume there are two boxes, B_1 and B_2. B_1 contains four slips numbered from 1 to 4, while box B_2 contains twenty slips numbered from 1 to 20. Each slip within a box is equally likely to be drawn. Let event E denote the selection of box B_1 and event F denote the selection of box B_2. One of the boxes is chosen at random, with equal probability ½ for each, and then a slip is drawn from the selected box. The observed outcome is that the number drawn is 3. Determine the Bayes factor in favour of event E given this observation.

(b) Suppose that $X_1, X_2, \dots, X_n$ is a random sample from $N(\theta, \sigma_0^2)$ where σ_0^2 is known. We wish to test the hypothesis $H_0: \theta = \theta_0$, assuming vague (non-informative) prior beliefs for θ. Using Lindley's procedure, find out the decision rule for accepting or rejecting the hypothesis H_0 at the 5% level of significance. Apply the rule to test $H_0: \theta = 1$ when $n = 16, \bar{x} = 2$ and $\sigma_0^2 = 1$ and conclude whether H_0 should be accepted or rejected.

(8, 7)

7. (a) The waiting time for a bus at a particular location and time of day is assumed to follow uniform distribution $U(0, \theta)$, where θ is unknown. It is desired to test the hypotheses $H_0: 0 \leq \theta \leq 15$ against $H_1: \theta > 15$. Based on information from similar routes, θ is assumed to follow $Pareto(5, 3)$ distribution. If waiting times of 10, 3, 2, 5, 14 are observed, calculate the posterior probability of each hypothesis and the posterior odds in favour of H_0. Also, compute the Bayes factor in favour of H_0, assuming that π_0 is the prior probability that H_0 is true. Comment on the result when both hypotheses are assumed to be equally likely a priori.

(b) Let $X \sim Bin(n, \theta)$. We wish to test the hypotheses $H_0: \theta = \frac{1}{2}$ against $H_1: \theta \neq \frac{1}{2}$. Assume that under H_1, the prior distribution of θ is uniform on $(0,1)$.

(i) Derive the Bayes factor in favour of H_0.

(ii) Obtain the posterior probability of H_0, where π is the prior probability of H_0.

(iii) Further, assuming that both hypotheses are equally likely a priori, compute the Bayes factor for $n = 5$ when (i) $x = 1$ and (ii) $x = 3$. Comment on how the Bayes factor reflects the strength of evidence in favour of H_0 for these given observations.

(7, 8)