(c) Let (X,d) a metric space and $A \subseteq X$. Define $f: X \rightarrow \mathbb{R}$ as
$$f(x) = d(x,A) = \inf\{d(x,y): y \in A\},\ x \in x$$
Prove that f is uniformly continuous over X.

5. (a) Show that the metric spaces (X, d) and (X, ρ), where
$$\rho(x,y)=\frac{d(x,y)}{1+d(x,y)};\ \forall x, y \in X$$
are equivalent.

(b) Let (X, d) be a disconnected metric space. Prove that there exists a continuous mapping of (X, d) onto the discrete two element space (X_0, d_0).

(c) Define a disconnected metric space. Give examples of a disconnected and a connected metric space.

6. (a) Let Y be a subset of a metric space (X,d). If (Y, d_Y) is compact then prove that Y is a closed subset of (X, d).

(b) Let f be a continuous function from a compact metric space (X, d_X) into a metric space (Y, d_Y). Then show that the range f(X) of f is also compact.

(c) Give examples of infinite subsets of $\mathbb{R}$ which are :

(i) Both connected and compact

(ii) Neither connected nor compact

(iii) Compact but not connected

(iv) Connected but not compact

[This question paper contains 4 printed pages.]

Your Roll No..............

Sr. No. of Question Paper : 9544 **K**

Unique Paper Code : 2353200013

Name of the Paper : Elements of Metric Spaces

Name of the Course : **B.A. / B.Sc. (Prog.) (DSE)**

Semester : VII

Duration : 3 Hours Maximum Marks : 90

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.

2. Attempt **all** the questions by selecting **two** parts from each question.

3. **All** questions are compulsory and carry equal marks.

4. Use of calculator is not allowed.

1. (a) Let (X, d) be a metric space. Define the function $d': X \times X \to \mathbb{R}$ by

$$d'(x,y) = \frac{d(x,y)}{1+d(x,y)}, \quad x, y \in X.$$

Show that d′ is a metric on X.

(b) Define the convergence of a sequence in a metric space (X, d). Prove that the limit of a sequence is unique.

(c) Find open and closed balls in the metric space $(\mathbb{R}^2, d_1)$ where

$d_1(x,y) = |x_1 - y_1| + |x_2 - y_2|, x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$,

with centre (0,0) and radius 1. Sketch it also.

2. (a) Define an open subset in a metric space (X, d). Show that every open ball in (X, d) is an open set. Is the converse true? Justify.

(b) For any two subsets A, B of a metric space (X, d). Prove that

(i) $(A \cap B)^\circ = A^\circ \cap B^\circ$

(ii) $A^0 \cup B^\circ \subset (A + B)^\circ$. Give an example to show that the equality may not hold in this case.

(c) For any two subsets A, B of a metric space (X, d). Prove that

(i) $\overline{A \cup B} = \bar{A} \cup \bar{B}$

(ii) $\overline{A \cap B} \subset \bar{A} \cap \bar{B}$. Give an example to show that equality may not hold in this case.

3. (a) Let (X, d) be metric space where $X = \mathbb{R}$ and d is usual metric. Let Y = [0,1) be a subspace of X. Show that $F_1 = [0,0.2]$ and $F_2 = [0.8,1)$ are closed in Y.

(b) Let (X, d) a metric space and Y be a subspace X. Let $z \in Y$ and $r > 0$ then

(i) Show that $S_Y(z, r) = S_X(z, r) \cap Y$.

(ii) Show that $\overline{S_Y(z,r)} = Y \cap \overline{S_X(z,r)}$.

(c) Prove that a complete subspace of any metric space is closed. Is converse true? Justify.

4. (a) Let (X, d_X) and (Y, d_Y) be two metric spaces and $A \subseteq X$ then prove that a function $f: A \to Y$ is continuous at $a \in A$. if whenever a sequence $\{x_n\}$ in A converges to 'a' then the sequence $\{f(x_n)\}$ converges to f(a).

(b) Let (X, d_X) and (Y, d_Y) be two metric spaces. A mapping $f: X \to Y$ is continuous if and only if $f^{-1}(F)$ is closed in X for all closed subsets F of Y.