(c) State the Singular Value Decomposition theorem for matrices and find the Singular Value Decomposition for the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}. \qquad (2+5.5)$$

[This question paper contains 8 printed pages.]

Your Roll No..............

Sr. No. of Question Paper : 9543 K

Unique Paper Code : 2353200012

Name of the Paper : Advanced Linear Algera

Name of the Course : B.A./B.Sc. (P)

Semester : VII – DSE

Duration : 3 Hours Maximum Marks : 90

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.

2. Attempt **all** questions by selecting **two** parts from each question.

3. **All** questions carry equal marks.

4. Use of Calculator is not allowed.

P.T.O.

1. (a) Let $P_1(\mathbb{R})$ denote the vector space of all real polynomials of degree less than or equal to 1. Let T be a linear operator on $P_1(\mathbb{R})$ defined by

$$T(p(x)) = p'(x),$$

the derivative of p(x). Let $\beta = \{1,x\}$ and $\beta' = \{1 + x, 1 - x\}$ be the ordered bases of $P_1(\mathbb{R})$. Find the change of coordinate matrix Q that change β'-coordinates into β-coordinates. Also, verify that

$$[T]_\beta = Q^{-1}[T]_{\beta'}Q. \qquad (2.5+5)$$

(b) Let $V = \mathbb{R}^3$ and define $f_1, f_2, f_3 \in V^*$ as follows :

$$f_1(x,y,z) = x + y,\ f_2(x,y,z) = y + z,\ f_3(x,y,z) = x + z.$$

Prove that $\{f_1, f_2, f_3\}$ is a basis for V^*, then find a basis for V for which it is the dual basis. (3+4.5)

(c) Let V denotes a finite-dimensional vector space. Define the annihilator S^0 of a subset S of V and prove that S^0 is a subspace of V^*. Further, if W_1 and W_2 are subspaces of V, then prove that

$$(W_1 + W_2)^0 = W_1^0 \cap W_2^0 \qquad (1.5+2+4)$$

(b) Find the best-fit linear function for the data $\{(-2,4), (-1,3), (0,1), (1,-1), (2,-3)\}$ using least squares approximation. Also, compute the error E. (5+2.5)

(c) Let $V = \mathbb{R}^3$, $u = (2,1,3)$ and $W = \{(x,y,z) \mid x + 3y - 2y = 0\}$. Find the orthogonal projection of the vector u onto the subspace W. (7.5)

6. (a) Let $V = M_{2\times2}(\mathbb{R})$ be the space of all 2×2 real matrices and T be the linear operator on V defined as $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ a & b \end{pmatrix}$. Check whether T is normal, self-adjoint, or neither. (7.5)

(b) State the Spectral Theorem. Let $V = \mathbb{R}^2$, $W = \text{span}(\{1,2\})$, and β be the standard ordered basis for V. Compute $[T]_\beta$, where T is the orthogonal projection of V onto W. (3+4.5)

(i) $\langle x,y + z\rangle = \langle x,y\rangle + \langle x,z\rangle$.

(ii) $\langle x,cy\rangle = \overline{c}\ \langle x,y\rangle$.

(iii) $\langle x,0\rangle = \langle 0,x\rangle = 0$.

(iv) $\langle x,x\rangle = 0$ if and only if $x = 0$.

(1.5+2+2+2)

5. (a) Let $P_2(\mathbb{R})$ be the space of all real polynomials of degree less than or equal to 2 with the inner product

$$\langle f,g\rangle = \int_0^1 f(t)g(t)\,dt.$$

Take $S = \{1,x,x^2\}$ and $h(x) = 1 + x$. Apply the Gram-Schmidt process to the set to obtain an orthonormal basis $B = \{v_1, v_2, v_3\}$ for $P_2(\mathbb{R})$. Also, find the sequence of inner products $\{\langle h,v_i\rangle\}_{i=1}^{3}$.

(5+2.5)

2. (a) Let $A = \begin{pmatrix} i & 0 \\ 1 & -2i \end{pmatrix} \in M_{2\times 2}(\mathbb{C})$. Determine all the eigenvalues A. For each eigenvalue λ of A, find the set of eigenvectors corresponding to λ. Also, find a basis for $\mathbb{C}^2$ consisting of eigenvectors of A. (2.5+5)

(b) For $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \in M_{2\times 2}(\mathbb{R})$, find an expression for A^n, where n is a positive integer. (7.5)

(c) Let $P_2(\mathbb{R})$ denote the vector space of all real polynomials of degree less than or equal to 2. Let T be a linear operator on $P_2(\mathbb{R})$ defined as

$$T(ax^2 + bx + c) = cx^2 + bx + a.$$

Prove that $P_2(\mathbb{R})$ is the direct sum of the eigenspaces of T. (7.5)

3. (a) Let T be a linear operator on a vector space V. Define a T-invariant subspace of V. Verify that for $V = P_3(\mathbb{R})$, and $T(f(x)) = f'(x)$, the subspace $W = P_2(\mathbb{R})$ is T-invariant of V. Further, show that the characteristic polynomial of the restriction T_W of T on W divides the characteristic polynomial of T. (1+2.5+4)

(b) State the Cayley-Hamilton Theorem for a square matrix A over a field F. Verify the same for the matrix :

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{pmatrix}.$$ (2+5.5)

(c) Let Tbe the linear operator on $\mathbb{R}^3$ defined by

$$T(a, b, c) = (3a + b - 2c, \ -a + 5c, \ -a - b + 4c).$$

Find a basis for each generalized eigenspace of T consisting of a union of disjoint cycles of generalized eigenvectors. Then find a Jordan canonical form J of T. (5.5+2)

4. (a) Consider the matrix :

$$A = \begin{pmatrix} 3 & -14 & 5 \\ 1 & -4 & 2 \\ 1 & -6 & 4 \end{pmatrix}$$

Find the minimal polynomial of the matrix A. What can we conclude about the diagonizability of the matrix A from its minimal polynomial? (5.5+2)

(b) In $\mathbb{C}^2$, show that $\langle x,y\rangle = xAy^*$ is an inner product, where

$$A = \begin{pmatrix} 1 & i \\ -i & 2 \end{pmatrix}.$$

Compute $\langle x,y\rangle$ for $x = (1 - i, 2 + 3i)$ and $(2 + i, 3 - 2i)$. (5.5+2)

(c) Let V be an inner product space over F. For $x, y, z \in V$ and $c \in F$, prove that :