[This question paper contains 2 printed pages.]

Your Roll No...............

Sr. No. of Question Paper : 9540 **K**

Unique Paper Code : 2353010018

Name of the Paper : DSE (iii) – Fundamentals of Topology

Name of the Course : **B.Sc. (Honors) Mathematics**

Semester : VII

Duration : 3 Hours Maximum Marks : 90

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Attempt all **six (06)** questions by doing any **two** parts from each question.
3. **All** questions carry equal marks.

Q1 a) Show that for any two points x and y of a metric space there exist disjoint open balls such that one is centred at x and the other at y. Also, show that the space of square summable sequences, l_2 space, is a metric space.

b) Show that convergence of a sequence in l_p space implies coordinate-wise convergence but the converse need not hold true.

c) Define the notions of an open cover and a subcover. Further, show that every second countable metric space is Lindelöf.

Q2 a) Show that the set of points of R at which the function $f : \mathbb{R} \to \mathbb{R}$ is not continuous forms a set of type F_σ. Deduce that there is no real-valued function defined on $\mathbb{R}$ that is continuous at each rational point and is discontinuous at each irrational point.

b) State and prove Tietze's Extension theorem.

c) Define and give one example of an arcwise connected metric space. Let (X, d) be a metric space and $x_0 \in X$ be any element. Prove that X is arcwise connected iff each $x \in X$ can be joined to x_0 by a path.

Q3 a) Prove that a subset Y of the metric space (X, d) is totally bounded iff every sequence in Y contains a Cauchy subsequence.

b) Define the notions of separable and Lindelöf metric spaces. Show that every Lindelöf metric space is separable.

c) Show that any complete metric space is of category II.

Q4. a) Define the notion of bases for a topological space. Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for the topologies $\Im$ and $\Im'$, respectively on X. Then prove the following are equivalent.

i) $\Im'$ is finer than $\Im$

ii) For each $x \in X$ and each basis element $B \in \mathcal{B}$ containing x, there is a basis element $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$.

b) If $\Im$ and $\Im'$ are topologies on X and $\Im'$ is strictly finer than $\Im$, what can you say about the corresponding subspace topologies on the subset Y of X Show that if A is a subspace of Y, then the product topology on $A \times B$ is same as the topology $A \times B$ inherits as a subspace of $X \times Y$.

c) Define Hausdorff Space. Given an example of such a space. Show that every finite point set in a Hausdorff space X is closed.

Q5. a) Let Y be a subspace of X. Then prove a set A is closed in Y if and only if it equals the intersection of a closed set of X with Y. Also, show that if U is an open subset of a topological space X and A is closed in X, then $U - A$ is open in X and $A - U$ is closed in X.

b) Let $Y = [-1,1]$ be a subspace of $\mathbb{R}_s$. Which of the following sets are open in Y and which are open in $\mathbb{R}$? Justify your answer.

(i) $A = \left\{x : \frac{1}{2} < |x| < 1\right\}$

(ii) $B = \left\{x : \frac{1}{2} < |x| \leq 1\right\}$

(iii) $C = \left\{x : \frac{1}{2} \leq |x| \leq 1\right\}$

c) Define a homeomorphism between two spaces X and Y. Give an example of a bijective function $f : X \rightarrow Y$ which is continuous but not a homeomorphism.

Q6. a) Show that a finite Cartesian product of connected spaces is connected.

b) Let $f : X \rightarrow Y$ be a bijective continuous map. Show that f is a homeomorphism if X is compact and Y is a Hausdorff space.

c) Which of the following subsets of $\mathbb{R}$ are connected? Justify your answer.

(i) $[-1,0) \cup (0,1]$

(ii) The set $\mathbb{Q}$ of rational numbers

(iii) The graph of $f(x) = \sin x$ over $[0, 2\pi]$

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