Your Roll No.......................

Sr. No. of Question Paper : **9538**

Unique Paper Code : 2353010016

Name of the Paper : Advanced Differential Equations

Name of the Course : B.Sc.(Hons.) Mathematics

Semester : VII

Time: 3 hours Maximum Marks: 90

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. All the six questions are compulsory.
3. Attempt any two parts from each question.
4. All questions carry equal marks.

1. (a) State and prove the Picard existence theorem for the solution of initial value problem $\frac{dy}{dx} = f(x,y), y(x_0) = y_0$

   (b) Using the Picard's method, solve the initial problem $y' = 1 + xy^2,\ y(0) = 0.$

   Perform only three iterations.

   (c) (i) State the equivalence between an IVP $\frac{dy}{dx} = f(x,y), y(x_0) = y_0$ and corresponding integral equation. Write the importance of this equivalence.

   (ii) Reduce the initial value problem

   $$y''' = x^2 + yy' + (y'')^2, y(0) = 1, y'(0) = -3, y''(0) = 0$$

   into a system of first order equations. Does there exist a unique solution of this initial value problem. Explain precisely why or why not?

2. (a) Prove that if $\boldsymbol{A}$ is an $n \times n$ constant matrix, then $\Psi(t) = e^{tA}$ is a fundamental matrix of the system $\frac{dY}{dt} = A(t)Y$ on J. Find the solution of the system satisfying $\phi(t_0) = E$ for any $t_0 \in J$.

   (b) Show that every nontrivial solution of the equation $y'' + e^x y = 0$ has infinitely many zeros on $(0, \infty)$, whereas a nontrivial solution of the equation $y'' - e^x y = 0$ has at-most one zero on the interval $(0, \infty)$.

   (c) (i) Prove: If Φ_j be the solutions of the system $\frac{dY}{dt} = A(t)Y$ satisfying $\Phi_j(t_0) = E_j$, where E_j are linearly independent vectors, then Φ_j are linearly independent.

   (ii) Find a fundamental matrix of the system $\frac{dY}{dt} = AY$ by using an exponential matrix for $A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$

3. (a) Prove that the Green's function for a boundary-value problem $L[y] = -f(x)$ in $[a, b]$, where $L \equiv \left(\frac{d}{dx}\right)\left[p(x)\left(\frac{d}{dx}\right)\right] + q(x)$ with homogeneous boundary conditions $a_1 y(a) + a_2 y'(a) = 0,\ b_1 y(b) + b_2 y(b) = 0$ is symmetric.

(b) Find the Green's function for the boundary-value problem

$$y'' = -x, y(0) = 0,\ y(1) = 0.$$

(c) Find the eigenvalues and eigenfunctions of the Sturm-Liouville problem

$$\frac{d}{dx}\left[x\frac{dy}{dx}\right] + \frac{\lambda}{x}y = 0, y'(1) = 0,\ y'(e^{2\pi}) = 0 \text{ where } \lambda > 0.$$

4. (a) Obtain the eigenvalues and eigenfunctions of the Sturm-Liouville system:

$$y'' + \lambda y = 0;\ y(1) = 0, \qquad y(0) + y'(0) = 0$$

(b) Determine the nature of the critical point of the following system and sketch the trajectories: $\frac{dx}{dt} = -3x + 2y,\ \frac{dy}{dt} = -2x$.

(c) Show that the critical point $(0,0)$ of the non-linear autonomous system

$$\frac{dx}{dt} = x + y + x^2 y$$

$$\frac{dy}{dt} = 3x - y + 2xy^3$$

is asymptotically stable.

5. (a) State and prove the Minimum principle for a harmonic function in two-dimension.

(b) State Dirichlet problem for upper half plane and find its solution using Fourier transform.

(c) State and prove the Helmholtz's first theorem.

6. (a) Prove that the solution of Dirichlet problem for Laplace equation depends continuously on the boundary conditions

(b) Using the method of Fourier transform, find the solution of the Dirichlet problem of Laplace equation for the upper half plane

$$u_{xx} + u_{yy} = 0, -\infty < x < \infty, x > 0,$$

$$u(0, y) = f(y), -\infty < x < \infty$$

with the condition that u is bounded as $x \to \infty$ and u, u_y vanish as $|y| \to \infty$

(c) Using Green's function, find the solution of three-dimensional Laplace equation for a semi-infinite space.