[This question paper contains 2 printed pages.]

Your Roll No...............

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Sr. No. of Question Paper : 5830

Unique Paper Code : 2352011102

Name of the Paper : DSC-2: Elementary Real Analysis

Name of the Course : **B.Sc. (H) Mathematics (NEP-UGCF 2022)**

Semester : I

Duration : 3 Hours

Maximum Marks : 90

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Attempt **all** questions by selecting **three** parts from each question.
3. The parts of each question are to be attempted together.
4. **All** questions carry equal marks.
5. Use of Calculator is not allowed.

1. (a) Find all $x \in \mathbb{R}$ that satisfy both $|2x-3| < 5$ and $|x+1| > 2$ simultaneously.

   (b) Let $U = \{x \in \mathbb{R} : 0 < x < 1\}$. Show that each element of U has some $\varepsilon-$ neighbourhood of it contained in U.

   (c) Let $S = \{x \in \mathbb{R} : x > 0\}$. Does S have lower bounds? Does S have upper bounds? Does inf S exist? Does sup S exist? Prove your statements.

   (d) Define supremum of a non-empty subset S of $\mathbb{R}$. Show with an example that a set can have a supremum without having a maximum. Justify your statements.

2. (a) Is the set $S = \{1 - \frac{1}{n} : n \in \mathbb{N}\}$ bounded? Find its supremum and infimum if they exist. Justify your statements.

   (b) Let A and B be two subsets of $\mathbb{R}$ which are both bounded above. Show that $A \cup B$ is a bounded above set. Find sup $(A \cup B)$ in terms of sup A and sup B. Prove your statements.

   (c) Suppose that A and B are non-empty bounded subsets of $\mathbb{R}$ that satisfy $a \leq b$ for all $a \in A$ and all $b \in B$. Prove that sup$A \leq$ inf B.

   (d) Show that a sequence in $\mathbb{R}$ can have at most one limit.

3. (a) Show that $\lim\limits_{n\to\infty} c^{1/n} = 1$ $for\ c > 0$.

   (b) If $\langle a_n \rangle$ and $\langle b_n \rangle$ converge to a and b respectively, prove that $\langle a_n b_n \rangle$ converges to ab.

   (c) State and prove the Squeeze Theorem.

   (d) Establish the convergence or the divergence of the sequence $\langle y_n \rangle$, where

$$y_n := \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \quad for\ n \in \mathbb{N}$$

4. (a) Prove that if $\langle x_n \rangle$ is a monotone increasing and bounded above sequence of real numbers, then it is convergent.

(b) If $\lim_{n\to\infty} x_n = x$, for any real sequence $\langle x_n \rangle$. Then show that $\lim_{n\to\infty} |x_n| = |x|$. Give an example to show that the convergence of $\langle |x_n| \rangle$ need not imply convergence of $\langle x_n \rangle$.

(c) Find limit inferior and limit superior of the following sequences:

(i) $\langle cos(\frac{n\pi}{2}) \rangle$

(ii) $\langle 2 + (-1)^{n+1} \rangle$

(d) State Cauchy Convergence Criterion for sequences. Hence show that the sequence $\langle a_n \rangle$, defined by $a_n = 1 + \frac{1}{2} + \cdots\cdots + \frac{1}{n}$, does not converge.

5. (a) Prove that for $a \neq 0$, the geometric series $\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + \cdots$ is convergent if and only if $|r| < 1$.

(b) Find the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$.

(c) Suppose $\sum a_n$ and $\sum b_n$ are non negative series. Let $\lim_{n\to\infty} \frac{a_n}{b_n} = L$. If $0 < L < \infty$, then prove that either both series converge or both series diverge.

(d) Check the convergence of the following series:

(i) $\sum_{n=1}^{\infty} \frac{n!}{e^n}$ (ii) $\sum_{n=1}^{\infty} sin^n\left(\frac{1}{n}\right)$

6. (a) Prove that the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ converges if and only if $p > 1$.

(b) State and prove the alternating series test.

(c) Define an alternating series. Check the convergence of the series $\sum_{n=1}^{\infty} (-1)^n n e^{-n}$.

(d) Determine for each of the following series whether the series converges absolutely, converges conditionally or diverges.

(i) $\sum_{n=2}^{\infty} (-1)^n \frac{1}{\ln n}$ (ii) $\sum_{n=1}^{\infty} (-1)^n \frac{\cos n\pi}{n}$