(b) Test for convergence

$$\int_0^\infty \frac{1}{\sqrt{x}e^x}dx$$

(c) Suppose $\int_a^b f$ is an improper integral of type I.

Show that if $\int_a^b |f|$ converges, then so does $\int_a^b f$.

(d) Show that $\int_0^1 t^{p-1}(1-t)^{q-1}dt$ converges for positive values of p and q.

[This question paper contains 8 printed pages.]

Your Roll No..............

Sr. No. of Question Paper : 5811 **K**

Unique Paper Code : 2352012302

Name of the Paper : Riemann Integration

Name of the Course : **B.Sc. (H) Mathematics**

Semester : III (DSC)

Duration : 3 Hours Maximum Marks : 90

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.

2. Attempt **all** questions, selecting any **three** parts from each question.

3. **All** questions carry equal marks.

4. Use of calculator is NOT allowed.

P.T.O.

1. (a) Calculate the upper and lower Darboux integrals for the function $f(x) = x^3$ on the interval $[0, b]$ and show that its integral value is $\frac{b^4}{4}$.

(b) Let f be a bounded function on [a, b]. If P is a partition of [a, b] then

$$m(b-a) \leq L(f,P) \leq U(f,P) \leq M(b-a)$$

where $M = \sup\{f(x): x \in [a, b]\}$ and $m = \inf \{f(x): x \in b]\}$. Verify the inequality for the function $f(x) = 3x + 1$, defined on [0,3] and the partition =

$P = \left\{0, \frac{1}{2}, 1, 2, 3\right\}$ of [0,3].

(c) Let f be a bounded function on [a, b]. Show that f is integrable over [a,b] if and only if for each $\in > 0$. there exists a partition P of [a,b] such that $U(f, P)-L(f,P) < \in$.

(b) Use cylindrical shell method to find the volume of the solid generated when the region enclosed by $y = x^3$, $y = 1$, $x = 0$ is revolved about the line $y = 1$.

(c) Find the exact arc length of the curve $x = \frac{y^4}{8} + \frac{1}{4y^2}$ from $y = 1$ to $y = 4$.

(d) Find the area of surface generated by revolving the curve

$$x = t,\ y = 2t^2,\ 0 \leq t \leq 1$$

about y-axis.

6. (a) Discuss the convergence or divergence of the following integrals.

(i) $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx$

(ii) $\int_{1}^{\infty} \frac{1}{x^3}\, dx$

(c) Show that if f and g are integrable functions over [a, b] and if f(x) $\le$ g(x) for $x \in [a, b]$, then

$\int_a^b f \le \int_a^b g$. Hence deduce that

$$\int_{-2\pi}^{2\pi} \left|x^2 \sin^8\left(e^x\right)\right| dx \le \frac{16\pi^3}{3}.$$

(d) Show that if f is a continuous real-valued function on [a, b] satisfying $\int_a^b f(x)g(x)\,dx = 0$ for every continuous function g on [a, b], then f(x) = 0 for all x in [a, b].

5. (a) Use disk and washer method to find the volume of the solid generated when the region enclosed by $y=\sqrt{x+1}$, $y=\sqrt{2x}$ and $y=0$ is revolved about x-axis.

(d) Define Riemann sum and Riemann integrability of a bounded function f on [a, b]. Let f be a bounded integrable function on [a, b] and $\langle S_n \rangle$ be a sequence of Riemann sums with corresponding partitions P_n satisfying $\lim_{n\to\infty} \text{mesh}(P_n) = 0$. Show that the sequence $\langle S_n \rangle$ converges to $\int_a^b f$.

2. (a) Show that if a function f is integrable on interval [a, b], then f is integrable on every interval $[c, d] \subseteq [a, b]$.

(b) Let f be a bounded function on [a, b]. If Q is a refinement of a partition P of [a, b], then show that $U(f,Q) - L(f,Q) \le U(f,P) - L(f,P)$.

(c) Prove that a bounded function fon [a, b] is Riemann integrable if it is Darboux integrable. Also compare their corresponding values of the integrals.

(d) Let f and F be functions on [0, 1]. given by $f(x) = \cos(x)$ for all $x \in [0, 1]$ and

$$F(x) = \begin{cases} 0, & \text{for } 0 \le x < \frac{\pi}{6} \\ 1, & \text{for } \frac{\pi}{6} \le x \le 1 \end{cases}$$

Show that f is F-integrable and $\int_0^1 f\, dF = \frac{\sqrt{3}}{2}$.

3. (a) Let f be an integrable function on [a,b] and there exists $\lambda > 0$ such that $|f(x)| \ge \lambda \ \forall \ x \in [a, b]$. Show that 1/f If is integrable over [a, b].

(b) Let f be a continuous function on [a, b] such that $f(x) \ge 0$ for all $x \in [a, b]$. Show that fa $\int_a^b f = 0$ if and only if $f(x) = 0$ for all $x \in [a, b]$.

(c) Let f be a function defined on [a, b] and $c \in (a, b)$ be any point. Show that if f is integrable on [a, c] and [c, b], then f is integrable on [a, b] and

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

(d) State and prove intermediate value theorem for integrals. Suppose f and g are continuous functions on [a, b] such that $\int_a^b f = \int_a^b g$. Prove there exists x in (a, b) such that $f(x) = g(x)$.

4. (a) Define piecewise continuous function. Show that every piecewise continuous function on [a, b] is integrable.

(b) State and prove Fundamental Theorem of Calculus I. Using the same, evaluate

$$\int_0^{\pi} x \cos x \, dx.$$