[This question paper contains 4 printed pages.]

Your Roll No...............

**Sr. No. of Question Paper : 5772** **K**

Unique Paper Code : 2352014701

Name of the Paper : Linear Analysis

Name of the Course : **Bachelor of Science (Honours Course) Mathematics**

Semester : VII

Duration : 3 Hours Maximum Marks : 90

**Instructions for Candidates**

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Attempt any **five** parts from Question No. **1**.
3. Attempt any **two** parts from Question No. **2** to **6**.

1. (a) Show that norm is a continuous mapping from $(X, \|\cdot\|)$ to $\mathbb{R}$.

(b) Fix $a = (\alpha_j) \in l^2$, define a linear functional $f: l^2 \to \mathbb{R}$ by $f(x) = \sum_{j=1}^{\infty} \alpha_j \xi_j$, where $x = (\xi_j)_{j=1}^{\infty}$. Find the norm of the functional f.

(c) Give definition of the matrix norm. Show that a norm on $M_n$ defined as

$$\|A\|_{\infty} = \max_{1 \le i,\, j \le n} |a_{ij}|, \quad \text{for} \quad A = (a_{ij}) \in M_n$$

is not a matrix norm.

(d) State parallelogram equality for an inner product space. Give an example to show that die norm on $l^p$ does not satisfy the parallelogram equality when $p \neq 2$.

(e) In an inner product space X, show that $y \perp x_n$ and $x_n \to x$ together imply $x \perp y$.

(f) Show that the graph G(T) of a linear operator T: $X \to Y$ is a vector subspace of $X \times Y$.

(g) Let X and Y be Banach spaces and T: $X \to Y$ an injective bounded linear operator. Show that $T^{-1}: \mathcal{R}(T) \to X$ is bounded if and only if $\mathcal{R}(T)$ is closed in Y. (3×5=15)

2. (a) Show that the normed space $(l^2, \|\cdot\|_2)$ is a Banach space.

(b) Give the definition of equivalent norms on a normed space X. Show that the below given norms on $\mathbb{R}^n$ are equivalent :

$$\|x\|_1 = |x_1| + |x_2| + |x_3| + \cdots + |x_n|,$$
$$\|x\|_2 = \sqrt{|x_1|^2 + |x_2|^2 + |x_3|^2 + \cdots + |x_n|^2},$$

where $x = (x_1, x_2, x_3, \ldots \ldots x_n) \in \mathbb{R}^n$.

(c) Prove that in a finite dimensional normed space X every linear operator on X is bounded. Is the finite dimensional condition necessary for the assertion? (7.5,7.5,7.5)

3. (a) Let X and Y be normed spaces and T: $\mathfrak{D}(T) \to Y$ be a linear operator. Then prove that T is continuous if and only if T is bounded.

(b) Prove that the dual space of $\mathbb{R}^n$ is isomorphic to $\mathbb{R}^n$.

(c) Find the norm of the linear functional $f : C[-1, 1] \to \mathbb{R}$ defined as

$$f(x) = \int_{-1}^{0} x(t)\,dt - \int_{0}^{1} x(t)\,dt.$$ (7.5,7.5,7.5)

4. (a) State the Cauchy-Schwarz inequality and, by using it prove the following :

Let $\{x_n\}$ and $\{y_n\}$ be two sequences in an inner product space such that $x_n \to x$ and $y_n \to y$, then show that $\langle x_n, y_n \rangle \to \langle x, y \rangle$.

(b) Let $\{e_k\}$ be any orthonormal sequence in an inner product space X. Show that for any $x, y \in X$,

$$\sum_{k=1}^{\infty} |\langle x, e_k \rangle \langle y, e_k \rangle| \le \|x\| \, \|y\| .$$

(c) Show that in an inner product space X, $x \perp y$ if and only if $\|x + \alpha y\| \ge \|x\|$ for all scalars $\alpha$.

(7.5,7.5,7.5)

5. (a) If z is any fixed element of an inner product space X, show that $f(x) = \langle x, z \rangle$ defines a bounded linear functional on X and $\|f\| = \|z\|$.

(b) Let $\{T_n\}$ be a sequence of bounded self-adjoint linear operators $T_n : H \to H$ on a Hilbert space H. Suppose that $\|T_n - T\| \to 0$, where $\|\cdot\|$ is the norm on the space B(H, H), the space of all bounded linear operators from H to H. Show that the limit operator T is a bounded self-adjoint linear operator on H.

(c) Let $T : H \to H$ be a bounded linear operator on a Hilbert space H.

(i) If T is self-adjoint, then show that $\langle Tx, x \rangle$ is real for all $x \in H$.

(ii) If H is complex and $\langle Tx, x \rangle$ is real for ail $x \in H$, then show that the operator T is self-adjoint.

(7.5,7.5,7.5)

6. (a) Let f be a bounded linear functional on a subspace Z of a normed space X. Then prove that there exists a bounded linear functional $\tilde{f}$ on X which is an extension of f to X and has the same norm,

$$|\tilde{f}\|_X = \|f\|_Z$$

where

$$\|\tilde{f}\|_X = \sup_{\substack{x \in X \\ \|x\|=1}} \left|\tilde{f}(x)\right|, \text{ and } \|\tilde{f}\|_Z = \sup_{\substack{x \in Z \\ \|x\|=1}} \left|\tilde{f}(x)\right|,$$

(and $\|f\|_Z = 0$ in the trivial case $Z = \{0\}$).

(b) Prove that every Hilbert space is reflexive.

(c) Define open mapping. Prove that a bounded linear operator T from a Banach space X onto a Banach space Y is an open mapping. Also, prove that if T is bijective then $T^{-1}$ is continuous. (7.5,7.5,7.5)