(c) Let the real-valued function f be continuous on the closed and bounded interval [a, b]. Prove that f is bounded and attains its maximum value and minimum value at points of [a, b]. (7.5)

[This question paper contains 8 printed pages.]

Your Roll No..............

Sr. No. of Question Paper : 13447 **K**

Unique Paper Code : 2354000013

Name of the Paper : Elements of Metric Spaces

Name of the Course : **Common Program Group (NEP-UGCF 2022) - GE**

Semester : VII

Duration : 3 Hours Maximum Marks : 90

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.

2. **All** questions are compulsory.

3. Attempt any **two** parts from each question.

4. In the question paper, given notations have their usual meaning unless and until stated otherwise.

1. (a) Define metric on a set. Show that the function d, defined on $\mathbb{R}$ by

$$d(x, y) = \frac{|x-y|}{1+|x-y|}$$ for each $x, y \in \mathbb{R}$, is a metric on $\mathbb{R}$. (2+5.5)

(b) Prove that in the discrete metric space (X, d) where d denotes the discrete metric, a sequence is convergent if, and only if, it is an eventually constant sequence. Is (X, d) a complete metric space? Justify your answer. (4+1+2.5)

(c) Prove that every convergent sequence is a Cauchy sequence but the converse is not true. State one condition under which a Cauchy sequence is convergent. (4+2+1.5)

(c) (i) Let (X, d) be a disconnected metric space. Prove that there exist two non-empty disjoint subsets A and B, both open in X with $X = A \cup B$.

(ii) Show that in R, the continuous image of an interval is an interval. (4,3.5)

6. (a) (i) Show that the interval I = (0,1) in the metric space ($\mathbb{R}$, d) where d denote the usual metric, is not compact.

(ii) Give an example of a non-empty set which is always compact irrespective of the metric imposed on it. (4,3.5)

(b) Let Y be closed subset of the compact metric space (X, d). Prove that (Y, d_Y) is compact. Further show that condition (X, d) is a compact metric space is not redundant. (4+3.5)

(c) Let (X, d_1) and (Y, d_2) be two metric spaces and let $f: X \to Y$. Then show that f is continuous on X if and only if $f(\overline{A}) \subseteq \overline{f(A)}$ for all subsets A of X. (7.5)

5. (a) (i) Given an example of a metric space whose every non-empty non-singleton set is disconnected.

(ii) Does there exist an onto continuous map $f: R \to R \setminus \{0\}$? (4,3.5)

(b) Let (X, d) be a metric space. Prove that (X, d) is connected if, and only if, the only continuous mappings of (X, d) into (X_0, d_0) are the constant mappings, where $X_0 = \{0,1\}$ and d_0 denotes the discrete metric on X_0, (7.5)

2. (a) (i) Let $X = \mathbb{R}$, d be the usual metric on $\mathbb{R}$. Prove that an open ball in $(\mathbb{R}, d)$ is a bounded open interval and conversely that any bounded open interval on the real line is an open ball.

(ii) Is it possible for a given ball not to have a unique centre or radius? Justify your answer. (4,3.5)

(b) Let $X = \mathbb{R}$, d be the usual metric and d_1 be the discrete metric on R. Let $A = \{0\} \cup \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\right\}$. Determine whether the set A is open or closed or none in the metric spaces (X, d) and (X, d_1). (5+2.5)

(c) Let Y be a subspace of a metric space (X, d). Prove that

(i) Every subset of Y that is open in Y is also open in X if, and only if, Y is open in X.

(ii) Every subset of Y that is closed in Y is also closed in X if, and only if, Y is closed in X.

(7.5)

3. (a) Prove the following :

(i) A function is continuous at every isolated point in its domain.

(ii) A function with discrete metric space as its domain is continuous.

(iii) The relation of isometric spaces is an equivalence relation. (2,2,3.5)

(b) Let (X, d_1) and (Y, d_2) be metric spaces and $A \subseteq X$, Then show that $f: X \to Y$ is continuous on X if and only if $f^{-1}(G)$ is open in X for all open subsets G of Y. (7.5)

(c) Prove that a uniformly continuous map is a continuous map but the converse is not true.

(3+4.5)

4. (a) Define a homeomorphism between metric spaces. Prove that the metric spaces $(0, \infty)$ and $\mathbb{R}$, equipped with the usual absolute metric are homeomorphic. (7.5)

(b) Show that the metric d_1 and d_∞ defined on $\mathbb{R}^n$ by

$$d_1(x,y) = \sum_{i=1}^{n} |x_i - y_i|, \quad x, y \in \mathbb{R}^n \text{ and}$$

$$d_\infty(x,y) = \max\{|x_i - y_i| : 1 \le i \le n\}, \quad x, y \in \mathbb{R}^n$$

are equivalent. (7.5)