[This question paper contains 4 printed pages.]

Your Roll No..............

Sr. No. of Question Paper : 5714 **K**

Unique Paper Code : 2352013501

Name of the Paper : Metric Spaces

Name of the Course : **B.Sc. (Hons) Mathematics (DSC-13)**

Semester : V

Duration : 3 Hours Maximum Marks : 90

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Attempt **all** question by selecting **two** parts from each question.
3. Part of the questions to be attempted together.
4. **All** questions carry equal marks.
5. Use of Calculator is not allowed.

1. (a) (i) Let $X = \mathbb{R}$ and for $x, y \in \mathbb{R}$, define $d(x, y)$ by

$$d(x,y)=\begin{cases}|x-y|+1, & \text{if exactly one among x and y is strictly positive}\\ |x-y|, & \text{otherwise}\end{cases}$$

Prove that (X, d) is a metric space. (4)

(ii) Let X denote the set of all Riemann integrable functions defined on $[a, b]$. For $f, g \in X$, define $d(f, g) = \int_a^b |f(x)-g(x)|\, dx$. Show that (X, d) is not a metric space. (3.5)

(b) Let X be the set of all continuous functions defined on [0, 2] equipped with metrics $d_\infty(f, g) = \sup_{x\in[0,2]}|f(x) - g(x)|$ and $d_1(f, g) = \int_0^2 |f(x)-g(x)|\, dx$, $f, g \in X$. Compute the distance $d_\infty(f, g)$ and $d_1(f, g)$ where

$$f(x)=\begin{cases}\sin x, & 0 \le x < \frac{\pi}{4} \\ \frac{1}{\sqrt{2}}, & \frac{\pi}{4} \le x \le 2\end{cases} \quad \text{and} \quad g(x)=\begin{cases}\cos x, & 0 \le x < \frac{\pi}{4} \\ \frac{1}{\sqrt{2}}, & \frac{\pi}{2} \le x \le 2\end{cases}.$$

(7.5)

(c) (i) Define complete metric space. Let $X = \mathbb{N}$ the set of natural numbers. Define $d(m,n) = \left|\frac{1}{m} - \frac{1}{n}\right|$; m, $n \in X$. Show that (X, d) is an incomplete metric space. (4)

(ii) Is the metric space (X, d) of the set X of rational numbers with usual metric d a complete metric space? Justify your answer. (3.5)

2. (a) (i) Define an open set in a metric space (X, d). Show that a subset G in a metric space (X, d) is open if and only if it is the union of all open balls contained in G. (5)

(ii) Let S(x, r) be an open ball in a metric space (X, d). Let A be a subset of X such that diameter of A, d(A) < r and $S(x, r) \cap A \neq \phi$. Show that $A \subseteq S(x, 2r)$. (2.5)

(b) (i) Let (X, d) be a metric space. Prove that the closed ball $\overline{S}(x,r)$, where $x \in X$ and $r > 0$, is a closed subset of X. (3.5)

(ii) Let $X = \mathbb{R}$ with usual metric. Let F = {1, 1/2, 1/3,...}. Find the derived set of F. (4)

(c) Let (X, d) be a metric space and A_1 and A_2 be subsets of X. Prove that

(i) $\overline{(A_1 \cup A_2)} = \overline{A_1} \cap \overline{A_2}$.

(ii) $\overline{(A_1 \cap A_2)} \subseteq \overline{A_1} \cap \overline{A_2}$.

Is the closure of the union of an arbitrary family of the subsets of X equal to the union of the closures of the members of the family? Justify your answer. (7.5)

3. (a) Let (X, d) be a metric space and $Z \subseteq Y \subseteq X$. If $cl_X Z$ and $cl_Y Z$ denotes the closures of Z in the metric spaces X and Y respectively, then show that $cl_Y Z = Y \cap cl_X Z$. (7.5)

(b) Let (X, d_X) and (Y, d_Y) be metric spaces and let $f: X \to Y$ be a function. Prove that the following statements are equivalent :

(i) f is continuous on X;

(ii) $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$ for all $B \subseteq Y$. (7.5)

(c) Let (X, d_X) and (Y, d_Y) be metric spaces. When will a function $f: X \to Y$ is said to be uniformly continuous? For any metric space (X, d) and $A \subseteq X$, let $g: X \to \mathbb{R}$ be defined as $g(x) = \inf\{d(x,y) : y \in A\}$. Prove that g is uniformly continuous mapping using ε-δ method. (7.5)

4. (a) Define equivalent metrics. Show that the metrics d_1, d_2, d_∞ defined on $\mathbb{R}^n$ by

$$d_1(x,y) = \sum_{i=1}^{n} |x_i - y_i|, \quad d_2(x,y) = \left(\sum_{i=1}^{n} (x_i - y_i)^2\right)^{\frac{1}{2}},$$

$d_\infty(x, y)$ — $\max\{|x_i - y_i|: 1 \le i \le n\}$ are equivalent metrics. (7.5)

(b) Define fixed point of the mapping T. Show that if $T: X \to X$ is a mapping and T^n has a fixed point $x_0 \in X$, then T also has a fixed point. Is the converse true? Justify your answer. (7.5)

(c) Prove that every isometry between metric spaces (X, d_X) and (Y, d_Y) is also a homeomorphism between (X, d_X) and (Y, d_Y). Let X be a metric space and $x_0 \in X$, C(X) denotes the space of all real valued, continuous bounded functions on X with uniform metric $d(f, g) = \sup\{|f(x) - g(x)|: x \in X\}$. Prove that the mapping $f_x(y) = d(y, x) - d(y, x_0)$ for $x, y \in X$ defines an isometry of X into C(X). (7.5)

5. (a) Let (X, d_X) be a connected metric space and f: $(X, d_X) \to (Y, d_Y)$ be a continuous mapping. Show that the space f (X) with the metric induced from Y is connected. (7.5)

(b) Let (X, dx) be a metric space. If every continuous function f: $(X, d_X) \to (\mathbb{R}, d)$ has the intermediate value property, prove that (X, d_X) is a connected metric space. (7.5)

(c) If every two points in a metric space X are contained in some connected subset of X, prove that X is connected. (7.5)

6. (a) Let Y be a subset of metric space (X, d). If (X, d_Y) is compact, prove that Y is a closed subset of (X, d). (7.5)

(b) If f is a continuous real-valued function on a compact metric space (X, d_X), show that f is bounded and attains its bounds, i.e., if M = sup f(X), m = inf f(X), there exist x and y in X such that f(x) = M and f(y) = m. (7.5)

(c) Let A be a compact subset of a metric space (X, d) and B be a close subset of X such that $A \cap B = \phi$. Show that d(A, B) > 0. (7.5)