[This question paper contains 4 printed pages.]

Your Roll No...............

Sr. No. of Question Paper	**:**	**9261**	**J**
Unique Paper Code	:	2354002003	
Name of the Paper	:	Elements of Real Analysis	
Name of the Course	:	**Common Program Group: GE**	
Semester	:	IV	

Duration : 3 Hours Maximum Marks : 90

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Attempt **all** questions by selecting **two** parts from each question.
3. Part of the questions to be attempted together.
4. **All** questions carry equal marks.

1. (a) Define maximum and minimum elements of a set in an ordered field. Prove that $\mathbb{N}$ has no maximum element. Does it have a supremum? A minimum? An infimum?

   (b) Prove that in an ordered field F, u is a lower bound of the set A if and only if $-u$ is an upper bound of the set $-A = \{-a: a \in A\}$.

   (c) Find all $x \in \mathbb{R}$ that satisfy the following inequalities:

   (i) $|3x + 5| > 2$

   (ii) $|x - 1| < |x|$

2. (a) In any ordered field F, prove the following properties :

(i) If $x, y > 0$, then $x < y$ if and only if $x^2 < y^2$.

(ii) $|x| > a$ if and only if $x > a$ or $x < -a$.

(b) In a complete ordered field F, let $A \subseteq B$ and A is nonempty. Prove that if B is bounded below, then so is A, and $\inf A \geq \inf B$.

(c) Show that a sequence cannot converge to more than one real number.

3. (a) Define a bounded sequence. Give an example of the following :

(i) a sequence which is bounded above but not bounded below.

(ii) a sequence which is bounded below but not bounded above.

(iii) an unbounded sequence.

(b) Use Algebra of Limits to find the following limits :

(i) $\lim\limits_{n\to\infty} \dfrac{3n^2-5}{2+7n^3}$

(ii) $\lim\limits_{n\to\infty} \dfrac{2n^2-n}{n^2-5n-7}$

(c) Prove that if $\{x_n\}$ is a bounded sequence (not necessarily convergent) and $\{y_n\}$ converges to 0, then the sequence $\{x_n y_n\}$ converges to 0. Hence, show that

$$\lim_{n\to\infty} \frac{\sin n}{n} = 0.$$

4. (a) If $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are sequences of real numbers such that $\lim\limits_{n\to\infty} a_n = L$, $\lim\limits_{n\to\infty} c_n = L$ and $a_n \leq b_n \leq c_n$ for all $n \in \mathbb{N}$, then $\lim\limits_{n\to\infty} c_n = L$.

(b) Let $\{x_n\}$ be a sequence of real numbers defined inductively by $x_1 = \frac{3}{2}$ and $x_{n+1} = 2 - \frac{1}{x_n}$, for all $n \in \mathbb{N}$. Prove that $\{x_n\}$ converges, and find its limit.

(c) Prove that the sequence $\{x_n\} = \left\{\frac{(-1)^n}{n+1}\right\}$ is a Cauchy sequence.

5. (a) State the comparison test for convergence of a series. Use it to test the convergence of the following series :

(i) $\sum_{n=1}^{\infty} \frac{1}{3^n + 1}$.

(ii) $\sum_{n=1}^{\infty} \frac{1}{n!}$.

(b) Prove that if Σa_n is a convergent series of positive terms, then series Σa_n^2 also converges. Show by an example that the converse may not be true.

(c) Determine the convergence of the series :

$$\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdot 8 \cdots (2n)} \cdot \frac{1}{n}.$$

6. (a) State the limit comparison test. Use it to determine whether the series is convergent/divergent:

$$\frac{1}{3 \cdot 7} + \frac{1}{4 \cdot 9} + \frac{1}{5 \cdot 11} + \frac{1}{6 \cdot 13} + \cdots.$$

(b) Determine the following series converges absolutely/conditionally/diverges :

(i) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin n}{n^2}$.

(ii) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$.

(c) State and prove the Cauchy's criterion for convergence of a series.