[This question paper contains 2 printed pages.]

Your Roll No...............

Sr. No. of Question Paper : 9259 **J**

Unique Paper Code : 2354000009

Name of the Paper : Abstract Algebra

Name of the Course : **COMMON PROG GROUP**

Semester : VI

Duration : 3 Hours Maximum Marks : 90

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. **All** the **six** questions are compulsory.
3. Attempt any **two** parts from each question.
4. Each part carries **7.5** marks.
5. Use of Calculator not allowed.

Q1 (a) Define $GL(2,\mathbb{F})$ and find the inverse of the element $\begin{bmatrix} 11 & 13 \\ 9 & 7 \end{bmatrix}$ in $GL(2,\mathbb{Z}_{17})$.

(b) Describe the symmetries of an equilateral triangle. Construct the corresponding Cayley Table.

(c) Prove that the set $G = \left\{ \begin{bmatrix} a & a & a \\ a & a & a \\ a & a & a \end{bmatrix} : a \in \mathbb{R} and\, a \neq 0 \right\}$ is a group under matrix multiplication.

Q2 (a) (i) Let G be a group. Prove that $(ab)^{-1} = b^{-1}a^{-1} \forall a, b \in G$.

(ii) Prove that a group G is abelian if $(ab)^{-1} = a^{-1}b^{-1} \forall a, b \in G$.

(b) Define the *order* of an element of a group. Define the group $U(n)$. Find the orders of all the elements of the group $U(15)$ under multiplication modulo 15.

(c) Define the *centralizer* of an element of a group. Define the group D_4. Find the centralizers of all the elements of D_4. Justify.

Q3.(a) Let $\mathbb{Z}$ denote the group of integers under addition. Is every subgroup of $\mathbb{Z}$ cyclic? Why? Describe all the subgroups of $\mathbb{Z}$. Let a be a group element with infinite order. Describe all subgroups of $< a >$.

(b) Let $\beta = (123)(145)$. Write β^{99} in disjoint cycle form. Also, in S_3, find elements α and β such that $|\alpha| = 2$, $|\beta| = 2$, and $|\alpha\beta| = 3$

(c) State the Lagrange's Theorem. Suppose that K is a proper subgroup of H and H is a proper subgroup of G. If $|K| = 42$ and $|G| = 420$, what are the possible orders of H?

Q4(a) Let H be a subgroup of a group G, then show that $aH = H$ if and only if $a \in H$ for every $a \in G$. Also, let $H = \{0, \pm 3, \pm 6, \pm 9, \dots\}$ be a subgroup of $\mathbb{Z}$, then find all the left cosets of H in $\mathbb{Z}$.

(b) Define a normal subgroup. Let $H = \{(1), (12)\}$. Is H normal in S_3.

(c) Consider the map $f: GL(2, \mathbb{R}) \to \mathbb{R}^*$ defined by $f(A) = det(A)$, where $det(A)$ represents the determinant of the matrix A. Show that f is a homomorphism. Also, find the $Kerf$.

Q5(a) For $aandb$ in a ring R, prove that:

(i) $a \cdot 0 = 0 \cdot a = 0,$

(ii) $a(-b) = (-a)b = -(ab),$

(iii) $(-1)a = -a,$

(iv) $(-1)(-1) = 1$, if R has a unity element 1.

(b) State and prove subring test. Let $R = \left\{\begin{bmatrix} a & a \\ b & b \end{bmatrix} | a, b \in \mathbb{Z}\right\}$. Prove or disprove that R is a subring of $M_2(\mathbb{Z})$.

(c) Prove that a finite integral domain is a field. Hence show that the set $\mathbb{Z}_3[i] = \{a + bi | a, b \in \mathbb{Z}_3\}$ is a field.

Q6 (a) State the ideal test.

(i) Prove that the intersection of any set of ideals of a ring is an ideal.

(ii) If A is an ideal of ring R such that either 1 belongs to A or Acontains a unit, then prove that $A = R$.

(b) (i) Determine all ring homomorphisms from $\mathbb{Z}_{12}$ to $\mathbb{Z}_{20}$.

(ii) Let $M = \left\{\begin{bmatrix} a & b \\ -b & a \end{bmatrix} | a, b \in \mathbb{R}\right\}$. Show that $\phi: \mathbb{C} \to M$ given by

$$\phi(a + bi) = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$ is a ring isomorphism.

(c) (i) Show that the correspondence $x \to 5x$ from $\mathbb{Z}_5$ to $\mathbb{Z}_{10}$ does not preserve addition.

(ii) Show that the correspondence $x \to 3x$ from $\mathbb{Z}_4$ to $\mathbb{Z}_{12}$ does not preserve multiplication.

(100)