[This question paper contains 4 printed pages.]

Your Roll No...............

Sr. No. of Question Paper : 9257 **J**

Unique Paper Code : 2353012005

Name of the Paper : Mathematical Modeling

Name of the Course : **B.Sc. (H) Mathematics**

Semester : IV – DSE

Duration : 3 Hours Maximum Marks : 90

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Each question contains **three** parts. Attempt **all** questions by selecting **two** parts from each question.
3. Parts of questions to be attempted together.
4. **All** questions carry equal mads.
5. Use of non-programmable scientific calculators is allowed.

1\. (a) Define Mathematical Modeling. Explain the modeling cycle process. A ball and bat together cost \$1.10. The bat costs \$1.00 more than the ball. Model this problem by using mathematical symbols and determine how much does the ball cost.

(b) What are the physical and mathematical equations? Find the dimensions of the parameters m, K, K_1, A and convert following differential equation into a dimensionless equation,

$m\frac{d^2x}{dt^2} + Kx + K_1x^3 = 0$. Initial conditions are $x(0) = A$, $\frac{dx(0)}{dt} = 0$.

(c) Consider the following susceptible-infected-recovered (SIR) model:

$$\frac{dS}{dt} = -\beta\sqrt{S}\sqrt{I}, \frac{dI}{dt} = \beta\sqrt{S}\sqrt{I} - \gamma\sqrt{I}, \frac{dR}{dt} = \gamma\sqrt{I},$$

where β and γ are constant parameters and initial subpopulations are given as $S(0) = S_0 > 0, I(0) = I_0 > 0, R(0) = 0$. By taking the first two equations of the model for analysis purposes, and assuming the transformation $u = \sqrt{S}, v = \sqrt{I}$, show that

$$v(t) = \left[\left(\sqrt{S_0} - \sqrt{S^*}\right)^2 + I_0\right]^{\frac{1}{2}} . \sin\left(\frac{\beta}{2}t + \phi_1\right), \text{ where } \phi_1 = \tan^{-1}\left(\frac{\sqrt{I_0}}{\sqrt{S_0}-\sqrt{S^*}}\right) \text{ and } S^* = \left(\frac{\gamma}{\beta}\right)^2.$$

2\. (a) In a particular epidemic model, where the infected individuals eventually recover, the population dynamics are governed by the following system of differential equations:

$$\frac{dS}{dt} = \frac{-\beta SI}{N}, \frac{dI}{dt} = \frac{\beta SI}{N} - \gamma I.$$

Given the parameter values $\beta = 2$ and $\gamma = 0.4$ and assuming that the total population N is 1000, in which initially there is only one infected individual but there are 999 susceptible individuals within the population, answer the following:

(i) Using the given parameters, calculate the basic reproduction number r_0.

(ii) What is the peak number of infected individuals at any time during the epidemic?

(b) Consider the susceptible-infected-recovered (SIR) model defined as:

$$\frac{dS}{dt} = -\beta\sqrt{S}\sqrt{I}, \frac{dI}{dt} = \beta\sqrt{S}\sqrt{I} - \gamma\sqrt{I}, \frac{dR}{dt} = \gamma\sqrt{I},$$

where β and γ are constant parameters, with initial subpopulations:

$$S(0) = S_0 > 0, I(0) = I_0 > 0, R(0) = R_0 \geq 0.$$

(i) Find the condition for epidemic occurrence by finding the basic reproduction number (r_0).

(ii) If the first integral expression in the (S, I) plane is given as $S(t) + I(t) - 2\sqrt{S^*S(t)} = I_0 + S_0 - 2\sqrt{S^*S_0}$, where $S^* = \left(\frac{\gamma}{\beta}\right)^2$, then find the expressions for the maximum infectives number (I_{max}), the final susceptible population number (S_∞), and the total count of the infectives population (I_{total}).

(c) Construct a Susceptible-Exposed-Infectious-Recovered (SEIR) model using a system of first order differential equations to describe the spread of disease. Divide the total population $N(t)$ and any time t into subpopulations according to disease status. Assuming a constant influx of individuals into the susceptible populations and a natural death rate affecting each subpopulation, as well as the exposure rate of susceptible individuals and recovery rate of infectious people, obtain the exact solution for the total population $N(t)$, and provide the discretized form of the model equations using the Nonstandard Finite Difference ($NSFD$) Scheme.

3\. (a) Find the nature and stability of the critical point (x^*, y^*) for the system of differential equations

(i) $\frac{dx}{dt} = x - 2y + 3, \frac{dy}{dt} = x - y + 2$

(ii) $\frac{dx}{dt} = 1 - y^2, \frac{dy}{dt} = x + 2y$

(b) Consider an undamped nonlinear spring-mass system. Let m denotes the mass of an object attached to a spring and let $x(t)$ denote the displacement of mass at time t from its equilibrium position. Let the force exerted by the spring on the mass is given by $F(x) = -kx + \beta x^3$, where $k > 0$.

(i) Find the expression for the total energy in the mass-spring system.

(ii) Discuss the stability of the system for both hard and soft spring cases separately.

(c) Find the critical point (x^*, y^*) for the system of differential equations

$$\frac{dx}{dt} = x - 5y - 5$$
$$\frac{dy}{dt} = x - y - 3$$

Using appropriate transformation, rewrite the system to the following form

$$\frac{du}{dt} = au + bv$$
$$\frac{dv}{dt} = cu + dv$$

Analyze the nature and stability of critical points of the original system and the transformed system.

4. (a) Let $x(t)$ and $y(t)$ be the population of competing species at time t. Consider the following system:

$$\frac{dx}{dt} = \frac{3}{2}x - x^2 - \frac{1}{2}xy$$
$$\frac{dy}{dt} = 2y - y^2 - \frac{3}{4}xy$$

(i) What is the nature of interaction between the two populations?
(ii) Find and characterize the system's critical points.

(b) Consider a second order differential equation $x'' + 4x' - 3(x')^2 + 4x^2 - 1 = 0$. Find the plane autonomous system from the given equation. Find and characterize the behavior of all critical points of the system.

(c) Determine the type and stability of the critical point (4,3) of the almost linear system.

$$\frac{dx}{dt} = 33 - 10x - 3y + x^2$$

$$\frac{dy}{dt} = -18 + 6x + 2y - xy.$$

5. (a) Using Monte Carlo Simulation, write an algorithm to calculate the area trapped between the curves $y = x^2$ and $y = x$.

(b) Explain the Linear Congruence Method of generating random numbers. Give drawbacks of this method. Find 12 random numbers using a=1, b=7, c=10 and 7 as the seed number. Was there cycling? If so, when did it occur.

(c) Using Monte Carlo Simulation, write an algorithm to calculate that part of the volume of the sphere $x^2 + y^2 + z^2 \leq 1$ that lies in first octant, $x > 0, y > 0, z > 0$.

6. (a) Use Simplex method to solve the following problem

$$Max.\ Z = 3x + 2y$$

s.to. $2x + y \le 40,$

$x + y \le 24,$

$2x + 3y \le 60,$

$x, y \ge 0.$

(b) Consider the following Linear Programming Problem:

$$Max. Z = 3x + y$$

s.to. $2x + y \le 6,$

$x + 3y \le 9,$

$x, y \ge 0.$

Using algebraic methods find the possible number of points of intersection in xy-plane. Are all the points feasible? If not, then how many are feasible and how many are infeasible. List the feasible extreme points along with the value of the objective function.

(c) Solve the following Linear Programming problem graphically.

$$Max. Z = 5x + 3y$$

s.to. $3x + 5y \le 15,$

$5x + 2y \le 10,$

$x, y \ge 0.$

Perform sensitivity analysis to find the range of values for the coefficient of x in the objective function for which the current extreme point remains optimal.