[This question paper contains 4 printed pages.]

Your Roll No...............

Sr. No. of Question Paper : 9255 **J**

Unique Paper Code : 2353200011

Name of the Paper : DSE : Introduction to'Mathematical Modeling

Name of the Course : **Bachelor of Arts (Programme) / Bachelor of Science (Physical Science and Mathematical Science)**

Semester : VI

Duration : 3 Hours Maximum Marks : 90

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Attempt **all** questions by selecting any **two** parts from each question.
3. **All** questions carry equal marks.
4. Use of non-programmable scientific calculator is allowed.

1. (a) Let x(t) denote the population of fish in a fish farm at any time t, and let h be the constant harvesting rate. Consider a fish population model with limited growth and harvesting, given by the differential equation : $\frac{dx}{dt} = rx\left(1-\frac{x}{K}\right)-h$, where r, k, and h are positive constants. Suppose $r = 1$, $k = 10$, $h = \frac{9}{10}$, and the initial population at time $t = 0$ is $x_0 > 0$. Using the data and model equation, investigate the behavior of the solution over time..

   (b) A large tank contains 100 litres of salt water, with s_0 kg of salt initially dissolved. Salt water flows into the tank at the rate of 10 liters per minute, and the concentration of salt in the incoming mixture, denoted by $c_{in}(t)$ (in kg of salt/litre), varies with time. The solution in the tank is assumed to be thoroughly mixed at all times, and salt water flows out at the same rate at which it flows in, so the total volume of the mixture in the tank remains constant. Using the balance law and formulating the corresponding word equation, find a differential equation for the amount of salt in the tank at any time t.

   (c) Consider the following system of differential equations :

$$\frac{dx}{dt} = I - k_1 x,\quad x(0)=0,$$

$$\frac{dy}{dt} = k_1 x - k_2 y,\quad y(0)=0,$$

where, k_1, $k_2 > 0$ represent the rates at which a drug moves between the two sequential compartments (GI-tract and the bloodstream), and I denotes the amount of drug released into the GI-tract in each time step. When a patient takes a course of a slowly dissolving cold pill, x(t) denotes the level of the drug in the GI-tract and y(t) denotes the level in the bloodstream at time t. Find the expressions for x(t) and y(t) that satisfy the given system of differential equations.

2. (a) Find the equilibrium solution(s) and determine whether each is stable or unstable for the following differential equations?

(i) $\frac{dy}{dt} = y - 1.$

(ii) $\frac{dC}{dt} = \frac{F}{V}c_i - \frac{F}{V}C$, where F, V, c_i are positive constants.

(b) Let N(t) denote the number of radioactive nuclei at any time t. Consider the radioactive decay model given by the differential equation : $\frac{dN}{dt} = -kN$, with initial condition $N(t_0) = n_0$, where $k > 0$ is a constant and n_0 is the number of nuclei at a time t_0:

(i) Solve this initial value problem.

(ii) Further, if the half-time is τ, find the expression of k in terms of τ.

(c) In a fish farm, fish are harvested at a constant rate of 2100 fish per week. The per capita death rate for fish is 0.2 fish per day per fish, and the per-capita birth rate is 0.7 fish per day per fish.

(i) Write a word equation describing the rate of change of the fish population. Hence, obtain a differential equation for the number of fish.

(ii) If the fish population at a given time is 240000, estimate the number of fish born in one week.

(iii) Determine whether there is any population level for which the fish population is in equilibrium.

3. (a) Consider two armies in a battle, the red army R(t) and the blue army B(t). Develop a system of differential equations for the number of soldiers in both the army in case both are randomly firing.

(b) Consider a disease where all those infected remain contagious for life. A model describing this, is given by the following system of differential equations :

$$\frac{dS}{dt} = -\beta SI,$$

$$\frac{dI}{dt} = \beta SI,$$

with $S(0) = S_0$, $I(0) = I_0$, $\beta > 0$.

(i) Use chain rule to find a relation between Susceptible S(t) and Effectives I(t).

(ii) Obtain and sketch phase plane curves.

(iii) Using the model, is it possible for all susceptible to become infected? Justify.

(c) Consider the system

$$\frac{dX}{dt} = d_1 X\left(1 - \frac{X}{K}\right) - c_1 YX,$$

$$\frac{dY}{dt} = c_2 YK - a_2 Y$$

for the dynamics of a predator Y(t) - preyX(t) model with density-dependent growth of prey and all parameters are positive constants.

(i) Find all equilibrium points.

(ii) Discuss the possibility of the extinction of one of the prey or predator.

4. (a) A model for spread of a disease, where all infectives I(t) recover from the disease and become susceptible S(t) again, is given by the pair of differential equations

$$\frac{dS}{dt} = -\beta SI + \alpha I$$

$\frac{dI}{dt} = \beta SI - \alpha I$, $S(0) = 762$, $I(0) = 1$, $\beta = .028$, $\alpha = .44$

(i) Use the chain rule to find a relationship between the number of susceptibles and the number of infectives.

(ii) Draw a sketch of typical phase-plane trajectories. Deduce the direction of travel along the trajectories, providing reasons.

(iii) Is the equilibrium point stable?

(b) Symbiosis is where two species interact with each other, in a mutually beneficial way. Starting with a compartmental diagram, formulate a differential equation model describing this process, based on the following. Assuming the per-capita death rate for each species to be constant, but the per-capita birth rate to be proportional to the density of the other species. In other words, the presence of other species is necessary for continued existence. Define all the parameters and variables involved in the model.

(c) Consider the following equations in battle model in case of aimed firing between Red army R(t) and Blue army B(t) :

$$\frac{dR}{dt} = -a_1 B,$$

$$\frac{dB}{dt} = -a_2 R, \text{ where } a_1 = 10^{-1},\ a_2 = 10^{-4},\ B(0) = 1000 \text{ and } R(0) = 100$$

(i) Use the chain rule to find a relationship between the number of soldiers in both the army.

(ii) Deduce the direction of travel along the trajectories, providing reasons.

5. (a) Using Monte Carlo Simulation, write an algorithm to find the volume under the surface
$x^2 + y^2 + z^2 \leq 4$ in the first octant $x > 0$, $y > 0$, $z > 0$.

(b) Use the middle-square method to generate 10 random numbers using a seed $x_0 = 1008$. Is there any drawback to this method? Justify.

(c) Use the transformed least-squares method to estimate the value of a in the model $y = ax^2$, given the data points :

x	0	1	2	3	4	5
y	3	6	8	11	13	14

6. (a) Use the linear congruence method $x_{n+1} = (a\, x_n + b) \bmod (c)$ to generate 10 random numbers using $a = 5$, $b = 1$, $c = 8$ and a seed $x_0 = 2$. Is there any drawback to this method? Justify.

(b) Use the least-squares method to estimate the coefficients a and b in the linear model $y = ax + b$, given the data points :

x	1	2	3	4	5	6
y	5	3	8	7	5	8

(c) Using Monte Carlo Simulation, write an algorithm to find the probability of getting a head when tossing an unfair coin, assuming the coin is biased according to the empirical distribution : P(head) = 0.6 and P(tail) = 0.4.