This question paper contains 4 printed pages]

Roll No. | | | | | | | | | | | |

S. No. of Question Paper : **7043**

Unique Paper Code : **2352203602**

Name of the Paper : **DSC-Elementary Mathematical Analysis**

Name of the Course : **NEP-UGCF B.A.(Prog.) with Mathematics as Major**

Semester : **VI**

Duration : **3 Hours** Maximum Marks : **90**

(Write your Roll No. on the top immediately on receipt of this question paper.)

Attempt any *two* parts from each question.

All questions are compulsory.

1. (*a*) Define the limit of a function in terms of $\in - \delta$. Use $\in - \delta$ definition, to show that $\lim\limits_{x \to 1}(4x + 1) = 5$. 7½

(*b*) Let $f : A \to \mathbf{R}$ where $A \subseteq \mathbf{R}$ and let x_0 be a cluster point of A. Then, prove that $\lim\limits_{x \to x_0} f(x) = L$ if and only if for each sequence $<x_n>$ in $A \setminus \{x_0\}$ that converges to x_0, the sequence $<f(x_n)>$ converges to L. 7½

(*c*) Give an example of a function $f : \mathbf{R} \to \mathbf{R}$ which is not continuous at every point of R, but $|f|$ is continuous everywhere on $\mathbf{R}$. Explain your answer with proper justification. 7½

P.T.O.

2. (a) Define the continuity of a function at a point in terms of $\in - \delta$. Use $\in - \delta$ definition, to prove that the function $f(x) = x^2 - 4$ is continuous at $x_0 = 2$. 7½

(b) Suppose $a \leq b$, and $f : [a, b] \to [a, b]$ is continuous. Then, show that there exists $c \in [a, b]$ such that $f(c) = c$. 7½

(c) Define uniform continuity of a function f on interval I. Use it to prove that the function $f(x) = 2x^2 - 5x + 2$ is uniformly continuous on the interval $[-3, 1]$. 7½

3. (a) Consider the function $f(x) = x^2$ on $[-1, 1]$ and the partition $P = \{-1, 0, 1/2, 1\}$. Find $\underline{S}(f, P)$ and $\overline{S}(f, P)$. Also, verify that $\underline{S}(f, P) \leq \overline{S}(f, P)$. 7½

(b) If Q is a refinement of a partition P, then prove that : 7½

(i) $\underline{S}(f, Q) \geq \underline{S}(f, P)$

(ii) $\overline{S}(f, Q) \leq \overline{S}(f, P)$.

(c) State Riemann's Criterion for integrability and show that $f(x) = 2x$ defined on $[0, b]$ is Riemann integrable. 7½

4. (a) Prove that the Dirichlet function $f : \mathbf{R} \to \mathbf{R}$ given by :

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$$

is not integrable on any closed interval $[a, b]$ where $a < b$. 7½

(b) Use sequential criterion for integrability to show that $f(x) = x^2 - 2x$ is integrable on $[0, 1]$, Also, find $\int_0^1 f$. 7½

(c) Prove that :

(i) If f is integrable on $[a, b]$ and $f(x) \geq 0$, $\forall\, x \in [a, b]$, then $\int_a^b f \geq 0$.

(ii) If f and g are integrable on $[a, b]$ and $f(x) \leq g(x)$, $\forall\, x \in [a, b]$, then $\int_a^b f \leq \int_a^b g$. 7½

5. (a) Define pointwise convergence and find the pointwise limit of the following sequence of functions :

(i) $<f_n>$ where $f_n(x) = \begin{cases} -1 & \text{if } -1 \leq x \leq -\dfrac{1}{n} \\ 0 & \text{if } -\dfrac{1}{n} < x < \dfrac{1}{n} \\ 1 & \text{if } \dfrac{1}{n} < x \leq 1 \end{cases}$ on $[-1, 1]$

(ii) $<g_n>$ where $g_n(x) = \dfrac{x^n}{1+x^n}$ on $[0, 2]$. 7½

(b) Let $f_n(x) = \dfrac{x}{n}$ on $\mathbf{R}$. Then show that the sequence $<f_n>$ is uniformly convergent on $[0, 1]$ but is not uniformly convergent on $\mathbf{R}$. 7½

(c) Show that $<e^{-nx^2}>$ is uniformly convergent on $[1, 4]$ and hence show $\lim_{n\to\infty} \int_1^4 e^{-nx^2} = 0$. 7½

6. (*a*) State Weierstrass M-Test to test the uniform convergence of a series of functions. Hence check the following series for uniform convergence :

(*i*) $\Sigma_{n=1}^{\infty} \dfrac{\sin nx}{n(n+1)}$ on $\mathbf{R}$

(*ii*) $\Sigma_{n=0}^{\infty} \dfrac{1}{1+n^4x^2}$ on [1, 2]. 7½

(*b*) Find the Radius of convergence and Interval of convergence for the following power series :

(*i*) $\Sigma_{n=0}^{\infty} \dfrac{n^2}{3^n}(x-5)^n$

(*ii*) $\Sigma_{n=0}^{\infty} \dfrac{n!}{5^n}x^n$. 7½

(*c*) Write Power Series expansion of $\tan^{-1} x$ and using Abel's theorem, show that : 7½

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots,$$