6. (a) What is random number ? Use the middle square method to generate 20 random numbers using $x_0 = 3043$.

(b) Use Monte Carlo simulation to approximate the area under $f(x) = \sqrt{x}$ over the interval $1/2 \le x \le 3/2$.

(c) Use the linear congrence method to generate 10 random numbers using a = 5, b = 1 and c = 8

***

500

[This question paper contains 8 printed pages]

Your Roll No. : ....

Sl. No. of Q. Paper : **7073** **I**

Unique Paper Code : 2353200011

Name of the Paper : Introduction to Mathematical Modeling

Name of the Course : **BA (P)/BSc (Physical Science & Mathematical Science) : DSE**

Semester : VI

Time : 3 Hours ***Maximum Marks : 90***

Instructions for Candidates :

(a) Write your Roll No. on the top immediately on receipt of this question paper.

(b) **All** the six questions are compulsory. Attempt any **two** Parts from each question. **All** questions carry equal marks. Use of Scientific Calculator is allowed.

1. (a) Consider the population growth model $\frac{dP(t)}{dt} = kP(t), k > 0$. If the population increase by 10% in each year, in how much years the population will be triple.

(b) Consider the population growth model $\frac{dX(t)}{dt} = kX(t), k > 0, X(0) = x_0$. Answer the following :

(i) Find all the equilibrium points, if any.

(ii) Determine whether the equilibrium points are stable or unstable.

(iii) If $x_0 = 100$, can the population ever reduce to zero, give reasoning behind your answer.

(c) Suppose S (t) denotes the amount of salt in a tank with capacity 200 liters. An incoming water at a rate 15 liter has the concentration of 2. Assuming that the inflow and outflow rate is same. Write the balance law that captures the rate of change of amount of salt in the tank and derive the differential equation. Further solve the differential equation.

2. (a) Consider the following system of equations

$$\frac{dx(t)}{dt} = -0.2x(t), \frac{dy(t)}{dt} = 0.2x(t) - 0.1y(t), x(0) = 100, y(0) = 0$$

5. (a) For the following data :

x	17	19	20	22	23	25	28	31	32	33	36	37	38	39	41
y	19	25	32	51	57	71	113	141	123	187	192	205	252	259	294

Test the exponential model $p = ae^{bt}$ by plotting the transformed data. Estimate the parameters a and b.

(b) For the following data set

x	1.0	2.3	3.7	4.2	6.1	7.0
y	3.6	3.0	3.2	5.1	5.3	6.8

Formulate the mathematical model that minimizes the largest deviation between the data and the line $y = ax + b$. Solve for the estimates of a and b.

(c) For the following data set :

$x\ (\times 10^{-3})$	5	10	20	30	40	50	60	70	80	90	100
$y\ (\times 10^{-5})$	0	19	57	94	134	173	216	256	297	343	390

Fit the data using the least squares method for $y = ax^2$.

(ii) Find all the euqilibrium solutions of the equations :

$$\frac{dX}{dt} = 3X - 2XY, \frac{dY}{dt} = XY - Y,$$

(c) For the standard SIR eqidemic model,

$$\frac{dS}{dt} = -\beta SI, \frac{dI}{dt} = \beta SI - \gamma I$$

(i) Show the equation for the phase – plane solution,

$I = -S + \frac{\gamma}{\beta}\log(S) + K$, where K is a constant of integration.

(ii) If s_f denotes the remaining number of susceptibles when there are no remaining infectives, show that $\frac{\beta}{\gamma} = \frac{\log(s_0/S_f)}{S_0 + i_0 - s_f}$, where s_0 and i_0 are the initial numbers of susceptible and infectives, respectively.

where x(t) and y(t) represent the amount of drug in GI – Track and blood at time t respectively. Solve the above system of equations.

(b) Consider the following density dependent population growth model with harvesting :

$$\frac{dx(t)}{dt} = rX\left(1 - \frac{X}{K}\right) - h, X(0) = x_0, \text{where}$$

$$r = 1, K = 10, h = \frac{9}{10}$$

Find the equilibrium points and examine whether the population increases or decreases when

(i) $x_0 < 1$

(ii) $1 < x_0 < 9$

(iii) $x_0 > 9$

(c) Suppose X (t) and Y (t) denotes the population of two adjacent countries at time t. Apart from natural birth and death let there be population migration from country A to country B, that is proportional to the population size of A. Draw compartment diagram along with balance law and derive the differential equations that capture the rate of change of populations.

3. (a) Consider the system

$$\frac{dX}{dt} = \beta_1 X\left(1 - \frac{X}{K}\right) - c_1 XY, \frac{dY}{dt} = c_2 XY - \alpha_2 Y,$$

for the dynamics of a predator – prey model, with density – dependent growth of the prey and all parameters positive constants. Find all the equilibrium points. How do they differ from those of the standard Lotka – Volterra system ?

(b) Consider the following model for two competing species, with densities,

$$\frac{dX}{dt} = \beta_1 X - c_1 XY - d_1 X^2, \frac{dY}{dt} = \beta_2 Y - c_2 XY - d_2 Y^2,$$

with parameter values

$$\beta_1 = 3, \beta_2 = 3, c_1 = 2, c_2 = 1, d_1 = 2 \text{ and } d_2 = 2.5$$

(i) What is the carrying capacity for each of the species, evaluated for given parameter values ?

(ii) Compare the given form of equations with density dependent growth of competing species and find K_1 and K_2 in terms of the given above parameters.

(c) Develop a model (a pair of differential equations) for a battle between two armies where both groups use aimed fire. Assume that the red army has a significant loss due to disease, where the associated death rate (from disease) is proportional to the number of soldiers in that army.

4. (a) When considering the growth of a population of small insects pests that interact with another population of beetle predators,

(i) make assumptions and construct a compartmental diagram for the predator – prey model and develop appropriate word equations for the each two populations the predator and the prey.

(ii) Formulate differential equations for the each two populations for word equations (i).

(b) (i) Use the chain rule to find I in terms of S, given that the initial number of susceptible is $S(0) = s_0$ and the initial number of contagious infective is $I(0) = i_0$.