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Roll No. | | | | | | | | | | | |

S. No. of Question Paper : **5803**

Unique Paper Code : **2352011202**

Name of the Paper : **Calculus-DSC 5**

Name of the Course : **Bachelor of Science (Honours Course) Mathematics NEP UGCF-2022**

Semester : **II**

Duration : **3 Hours** Maximum Marks : **90**

(Write your Roll No. on the top immediately on receipt of this question paper.)

Attempt *all* questions by selecting *three* parts from each question.

All questions carry equal marks.

Use of calculator is not allowed.

1. (*a*) Using $\varepsilon - \delta$ definition of limit, show that :

$$\lim_{x \to c} \frac{1}{x} = \frac{1}{c},$$

if $c > 0$.

(*b*) Show that if $f : (a; \infty) \to \mathbf{R}$ $(a > 0)$ is such that

$$\lim_{x \to \infty} xf(x) = \text{L},$$

where $\text{L} \in \mathbf{R}$, then

$$\lim_{x \to \infty} f(x) = 0.$$

P.T.O.

(c) Let $A \subset \mathbf{R}$ and $f : A \to \mathbf{R}$ has a limit at $c \in \mathbf{R}$, where c is a cluster point of A. Then prove that f is bounded on some neighbourhood of c.

(d) Give example of functions f and g such that f and g do not have limits at a point c, but both $f + g$ and fg have limits at c.

2. (a) If f is continuous on a closed interval $[a, b]$, then prove that f is uniformly continuous on $[a, b]$.

(b) Show that the function $f(x) = x^2$ is uniformly continuous on $[-4, 4]$.

(c) Show that :

$$\lim_{x \to \infty} \frac{x+2}{\sqrt{x}} \, (x > 0)$$

does not exist in $\mathbf{R}$.

(d) Use $\varepsilon - \delta$ definition of limit to show that :

$$\lim_{x \to 6} \frac{x^2 - 3x}{x+3} = 2.$$

3. (a) Let

$$f(x) = \begin{cases} x^2, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

Calculate f' on $\mathbf{R}$. Is f' continuous as well as differentiable on $\mathbf{R}$? Justify your answer.

(b) Let $f(x) = \sin |x|$, where $x \in \mathbf{R}$. Determine the set of points where f is not differentiable. Justify your answer.

(c) If $f : \mathbf{R} \to \mathbf{R}$ is differentiable at $a \in \mathbf{R}$, then show that :

(i) $$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = f'(a),$$

(ii) $$\lim_{h\to 0} \frac{f(a+h)-f(a-h)}{2h} = f'(a).$$

(d) Suppose f is defined on an open interval (a, b) Let $x_0 \in (a, b)$. If f assumes its maximum at x_0 and if f is differentiable at x_0, then show that $f'(x_0) = 0$.

4. (a) State and prove Rolle's theorem.

(b) Let f be defined on $\mathbf{R}$ and suppose that $|f(x) - f(y)| \le |x - y|^3$ for all $x, y \in \mathbf{R}$. Prove that f is a constant function on $\mathbf{R}$.

(c) Show that $\sin x \le x$ for all $x \ge 0$.

(d) State Intermediate value theorem for derivatives. Suppose f is differentiable on $\mathbf{R}$ and $f(0) = 1$, $f(1) = 2$ and $f(2) = 2$.

(i) Show that $f'(x) = \frac{1}{2}$ for some $x \in (0, 2)$,

(ii) Show that $f'(x) = \frac{1}{3}$ for some $x \in (0, 2)$.

5. (a) If

$$y = \log\left(x + \sqrt{1 + x^2}\right),$$

show that

$$(1 + x^2)y_{n+2} + (2n + 1)xy_{n+1} + n^2 y_n = 0.$$

(b) Trace the curve :

$$4x^2 = y^2(x^2 - 4).$$

(c) Determine the intervals of concavity and points of inflexion (if any) of the curve :

$$y = 2x^4 - 3x^2 + 2x + 1.$$

(d) State Taylor's Theorem. Find the Taylor's series expansion of $\sin 2x$ about $x = 0$.

6. (a) Trace the curve :

$$r = 2 \sin 2\theta.$$

(b) Determine the position and nature of the double points of the curve :

$$x^3 + 2x^2 + 2xy - y^2 + 5x - 2y = 0.$$

(c) Find the horizontal asymptotes of the graph of the function given by

$$y = x^5\left[\sin\frac{1}{x} - \frac{1}{x} + \frac{1}{6x^3}\right].$$

(d) Find the equations of tangents at origin to the curve :

$$x^4 + y^4 = a^2(x^2 - y^2).$$