(b) The following data is given for the variable x and y

x	1	2	3	4	5
y	6	12	36	65	95

Formulate the least squares mathematical model of the form $y = c_1x^2 + c_2x + c_3$ that minimizes the sum of the squared deviations between data and the model.

(c) The following data is given

$x\ (\times 10^{-3})$	5	10	20	30	40	50	60	70	80	90	100
$y\ (\times 10^{-5})$	0	19	57	94	134	173	216	256	297	343	390

Fit the data using the least squares method for $y = ax$.

6. (a) What is random number? Use the middle square method to generate 15 random numbers using $x_0 = 3043$.

(b) Using Monte Carlo simulation, write an algorithm to calculate the volume trapped between the two paraboloids $z = 8 - x^2 - y^2$ and $z = x^2 + 3y^2$. Note that the two paraboloids intersect on the elliptic cylinder $x^2 + 2y^2 = 4$.

(c) Use the linear congruence method to generate 20 random numbers using a = 5, b = 3 and c = 16.

(500)

[This question paper contains 8 printed pages.]

Your Roll No..............

Sr. No. of Question Paper : 3721 **J**

Unique Paper Code : 2353200011

Name of the Paper : Introduction to Mathematical Modeling

Name of the Course : **BA(P)/BSc (Physical Science & Mathematical Science): DSE**

Semester : VI

Duration : 3 Hours Maximum Marks : 90

Instructions for Candidates

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2. All the **six** questions are compulsory.
3. Attempt any **two** parts from each question.
4. **All** questions carry equal marks.
5. Use of Scientific Calculator is allowed.

1. (a) Let N(t) denotes the amount of certain radioactive material at time t, suppose that this radioactive material decay at a rate of 0.25 per unit of time. If the initial amount of material is 100, in how much time the radioactive material will reduce to half.

(b) Consider the following differential equation:

$$\frac{dC(t)}{dt} = \frac{F}{V}C_{in} - \frac{F}{V}C(t),\ C(0) = c_0$$

where C(t) is denotes the concentration of pollutant in lake at time t, F is the constant flow rate, V is the volume of lake and C_{in} is the concentration of pollutant in the incoming water. Answer the followings :

(i) Find the equilibrium solutions.

(ii) Identify the stable and unstable equilibrium solutions.

(iii) If C_{in} equals to zero, plot a representative graph of C(t) over a long period of time.

(c) Suppose S(t) denotes the amount of salt in a tank with capacity 100 litres. An incoming water at a rate 10 litre has the concentration of salt C (constant). Assuming that the inflow and outflow

(c) Considering two models for continuous vaccination of the susceptible population; the first assumes the vaccine gives perfect life-long protection and the second assumes the vaccine is only temporary. Ignore any births and deaths.

(i) Formulate a model for the spread of a disease where life-long immunity is attained after catching the disease. The susceptible are continuously vaccinated against the disease at a rate a proportion v per unit time of susceptible are vaccinated. To track the number vaccinated so here need to have an additional variable V(t).

(ii) Assume the vaccine is only partial effective, where vaccinated individuals return to the susceptible state after being protected by vaccination for an average time μ^{-1}.

5. (a) The following data represent the growth of a population of fruit flies over a 6-week period

t (days)	7	14	21	28	35	42
P (number of files)	8	41	133	250	280	297

Test the straight line model $P = c_1 t$ by plotting an appropriate set of data. Estimate the parameter c_1.

where this model can only predict outbreaks and not deal with the situation where cholera has become endemic.

(i) To model endemic cholera, modify this model to account for susceptible births and also incorporate natural deaths of all populations. Give the differential equations for R(t), the number of recovered. Assume per-capita birth rate b for the whole population. Assume per-capita death rate a is the same for susceptible, infected and recovered.

(ii) When is the total population constant? Find the disease-free equilibrium for the case where the deaths are balanced exactly with the births.

(b) (i) Find all the equilibrium solutions of the equations :

$$\frac{dX}{dt} = Y - 2XY, \frac{dY}{dt} = XY - Y^2$$

(ii) For the Lotka-Volterra predator-prey model

$$\frac{dX}{dt} = \beta_1 X - c_1 XY, \frac{dY}{dt} = \alpha_2 Y + c_2 XY,$$

Use the chain rule to find the equation relating to X and Y.

rate is same. Write the balance law that captures the rate of change of amount of salt in the tank and derive the differential equation. Further solve the differential equation.

2. (a) Consider the following system of equations

$$\frac{dx(t)}{dt} = -k_1 x(t), \quad \frac{dy(t)}{dt} = k_1 x(t) - k_2 y(t), \quad x(0) = x_0, \quad y(0) = 0$$

Where x(t) and y(t) represent the amount of drug in GI-Track and blood at time t respectively. Solve the above system of equations and write the behaviour of solution as t tends to infinity.

(b) Solve the following density dependent population growth model:

$$\frac{dX(t)}{dt} = r X \left(1 - \frac{X}{K}\right), X(0) = x_0$$

If $R(X) = r\,(1 - \frac{X}{K})$, gives an interpretation of R(X), by comparing the given differential equation with the unrestricted population growth model $\frac{dX(t)}{dt} = r\,X(t)$.

(c) A mouse population, initially consisting of 1,000 mice, has a per-capita birth rate of 8 mice per month (per mouse) and a per-capita death rate of 2 mice per month (per mouse). Also, 20 mouse traps are set each week and they are always filled. Draw a compartment diagram and write a word equation describing the rate of change in the number of mice and then formulate a differential equation for the population with an initial condition.

3. (a) Some diseases are spread largely by carriers, individuals who can transmit the disease but who exhibit no overt symptoms.: Let x and y, respectively, denote the proportion of susceptibles and carriers in the population. Suppose that carriers are identified and removal from the population at a rate β, so that $dy/dt = -\beta y$. Suppose also that the disease spread at a rate proportionate to the product of x and y, thus $dx/dt = -\alpha xy$,

(i) Determine the proportions of carriers at any time t, where $y(0) = y_0$.

(ii) Use the result of part (i) to find the susceptibles at time t, where $x(0) = x_0$.

(b) Consider the endemic infectious disease model with natural per-capita birth rare equal to natural per-capita death rate, a = b,

$$\frac{dS}{dt} = aN - \beta SI - as,\ \frac{dI}{dt} = \beta SI - \gamma I - aI,\ \frac{dR}{dt} = \gamma I - aR,$$

where $N(t) = S(t) + I(t) + R(t)$.

(i) Show there are only two equilibrium points, one corresponding to a disease-free equilibrium and one corresponding to an endemic equilibrium.

(ii) Write the endemic equilibrium solution in terms of the basic reproduction number $R_0 = \beta N/(\gamma + a)$ for this system. Discuss the cases $R_0 > 1$ and $R_0 < 1$.

(c) If the basic epidemic model is modified to include density-dependent disease transmission, then show that the resulting differential equations are

$$\frac{dS}{dt} = -p\frac{c(N)}{N}SI,\ \frac{dI}{dt} = p\frac{c(N)}{N}SI - \gamma I,$$

and the contact rate function

$$c(N) = \frac{c_m N}{K(1-\varepsilon) + \varepsilon N},$$

where N = S + I, p is a constant, ε is a positive constant between 0 and 1, and K is also a positive constant.

4. (a) Considering cholera model

$$\frac{dS}{dt} = -c\frac{BS}{K_{50}+B}, \frac{dI}{dt} = c\frac{BS}{K_{50}+B} - \gamma I, \frac{dB}{dt} = eI + (n_b - m_b)B,$$

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