6. (a) What is the random number ? Generate 25 random numbers using the seed $x_0 = 653217$.

(b) Use Monte – Carlo simulation to approximate the area under the curve $f(x) = \sqrt{x}$, over the interval $1/2 \le x \le 3/2$

(c) Find the area enclosed between the curves $y = x^2$ and $y = 6 - x$ within the region bounded by $x = 0$ and $y = 0$.

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[This question paper contains 8 printed pages]

Your Roll No. : ....

Sl. No. of Q. Paper : **5905** **I**

Unique Paper Code : 2353200011

Name of the Paper : Introduction to Mathematical Modeling

Name of the Course : **BA (P)/BSc (Physical Science & Mathematical Science) : DSE**

Semester : VI

Time : 3 Hours ***Maximum Marks : 90***

Instructions for Candidates :

(a) Write your Roll No. on the top immediately on receipt of this question paper.

(b) **All** the six questions are compulsory. Attempt any **two** Parts from each question. **All** questions carry equal marks. Use of Scientific Calculator is allowed.

1. (a) Consider the radioactive material decay problem. Let N(t) denotes the amount of radioactive material at time t, and k denotes the rate of decay per unit of material at per unit of time. Derive the differential equation that captures the rate of change of N(t). If the half life of material is α, Find k in terms of α.

P.T.O.

(b) A public bar opens at 6 p. m. and is rapidly filled with clients of whom the majority are smokers. The bar is equipped with ventilators that exchange the smoke – air mixture with fresh air. Cigarette smoke contains 4% carbon monoxide and a prolonged exposure to a concentration of more than 0.012% can be fatal. The bar has a floor area of 20m by 15 m, and height of 4 m. It is estimated that smoke enters the room at a constant rate of 0.006 m^3/min and that the ventilators remove the mixture of smoke and air at 10 times the rate at which smoke is produced.

Formulate the differential equation for the changing concentration of carbon monoxide in the bar over time. By solving the equation, establish at what time the lethal limit will be reached.

(c) Suppose a tank has constant volume V, which is continuously well mixed. Let C(t) be the concentration of salt at time t and F is the flow rate in the tank. Let C_{in} be the concentration of salt in the incoming water and c_0 be the initial level of salt in the tanks. Answer the following :

(i) Draw the compartment diagram and the word equations describing balance law.

5. (a) The following data represents the growth of a population of fruit flies over a 6 – week period :

t (days)	7	14	21	28	35	42
P (number of files)	8	41	133	250	280	297

Test the exponential model $P = ae^{bt}$ by plotting the transformed data (Semi – log plot). Estimating the parameters a and b.

(b) For the following dataset :

x	1.0	2.3	3.7	4.2	6.1	7.0
y	3.6	3.0	3.2	5.1	5.3	6.8

Formulate the mathematical model that minimizes the largest deviation between the data and the line $y = ax + b$. Determine the estimates for parameters a and b.

(c) The following data is provided :

x	2.5	3.0	3.5	4.0	4.5	5.0	5.5
y	4.32	4.83	5.27	5.74	6.26	6.79	7.23

Fit the data using the least squares method for $y = ax^2$.

(ii) Find the equilibrium solutions of the differential equations

$$\frac{dX}{dt} = \beta_1 X - c_1 XY, \frac{dY}{dt} = -\alpha_2 Y + c_2 XY$$

(c) A following mathematical model for disease, where all infectives die from the disease with death rate α, and where the transmission is assumed to be frequency dependent so the contract rate is constant, The differential equations are :

$$\frac{dS}{dt} = -pc\frac{I}{N}, \frac{dI}{dt} = pc\frac{I}{N} - \alpha I,$$

where N = S + I is the total population size and c (constant) per – capita contact rate and p is the constant probability of a contact resulting in an infection.

(i) Write the system in terms of the variables N and I and hence show that :

$$\frac{dI}{dN} = \frac{pc}{\alpha}\frac{I}{N} + \left(1 - \frac{pc}{\alpha}\right)$$

(ii) Solve the differential equation in (i) to obtain $I = N + KN^{pc/\alpha}$, where K is the arbitrary constant.

(ii) Derive a differential equation that captures the rate of change of C(t) and solve it.

2. (a) Consider the drug assimilation model in which when a course of pill is taken at a constant rate I per unit of time by an individual, it first goes to the gastrointestinal tract (GI – tract), and then diffused in the bloodstream. Let X(t) and Y(t) represents the amount of pill in the GI – Track and blood at time t. Derive differential equations that captures the rate of change of X(t) and Y(t) in the case of a course of pill, where a constant amount I is taken per unit of time. Further plot a representative graph for X(t) and Y(t) that shows the amount of drug in the GI Track and blood over a period of time.

(b) Let X(t) denotes the population at time t. By taking suitable parameters derive the Logistic Equation with harvesting that captures the density dependent growth rate.

(c) Suppose X(t) and Y(t) denotes the population of two adjacent countries at time t. Apart from natural birth and death let there be population migration from one country to other country that is proportional to the population size. Draw compartment diagram along with balance law and derive the differential equations that capture the rate of change of populations.

3. (a) (i) Find the basic reproduction number, R_0, for the SIR model,

$$\frac{dS}{dt} = -\beta SI, \frac{dI}{dt} = \beta SI - \gamma I$$

(ii) Also find the conditions for disease outbreak to die out and endemic according to R_0 and justify it with differential equation for I.

(b) The equations for the SIR epidemic model with frequency dependent transmission,

$$\frac{dS}{dt} = -\beta_f S\frac{I}{N}, \frac{dI}{dt} = \beta_f S\frac{I}{N} - \gamma I,$$

Where derived for the numbers of susceptible and infectives.

(i) What are the equations for the density susceptible and infectives ?

(ii) Also deal with the density – dependent transmission case.

(c) Let X (t) and Y (t) denote the two populations densities, where t is time. Here, birth and death rates are considered.

(i) make some assumptions and construct compartmental diagram for the competing species and formulate differential equations for each of the populations.

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(ii) extend the competing model (i) to account for logistic growth in both species in the absence of the other spices.

4. (a) When considering a destructive competition or battle between two opposing groups such as battles between two hostile insect groups, athletic teams, or human armies,

(i) make some assumptions and determine the appropriate input – output diagram and associated word equations for the number of soldiers in both the red and blue armies.

(ii) formulate differential equations for each of the two armies, the red and the blue of word equations (i).

(b) (i) Considering the model

$$\frac{dS}{dt} = -\beta SI, \frac{dI}{dt} = \beta SI - \gamma I,$$

Use the chain rule to find 'I' in terms of S, given that the initial number of susceptible is s_0 and the initial number of contagious infectives is i_0.