This question paper contains 8 printed pages]

Roll No. [| | | | | | | | | | |]

S. No. of Question Paper : **7603**

Unique Paper Code : **2353200011**

Name of the Paper : **DSE : Introduction to Mathematical Modeling**

Name of the Course : **Bachelor of Arts (Programme)/Bachelor of Science (Physical Science and Mathematical Science)**

Semester : **VI**

Duration : **3 Hours** Maximum Marks : **90**

(Write your Roll No. on the top immediately on receipt of this question paper.)

Attempt *all* questions by selecting any *two* parts from each question.

All questions carry equal marks.

Use of non-programmable scientific calculator is allowed.

1. (*a*) Consider a population growth model given by the following differential equation :

$$\frac{dP}{dt} = aP - bP^2,$$

where a and b are constants. In the model, bP^2 represents a death term due to overcrowding, which is assumed to be proportional to P^2 because of intra-population interactions :

(*i*) Find all the equilibrium points of this model.

(ii) Are there any conditions on the parameters a and b for the equilibrium population to remain positive ?

(iii) Also, determine the stability of each non-negative equilibrium.

(b) Define and explain the following :

(i) Balance law with an example.

(ii) Residence time for a radioactive particle.

(iii) Equilibrium solution for the system of the form :

$$\frac{dy}{dx} = f(y),$$

with an example.

(c) Let $y(t)$ denote the population of fish in a fish farm at any time t, and let h be the constant harvesting rate. Consider a fish population model with limited growth and harvesting, given by the differential equation :

$$\frac{dy}{dt} = ry\left(1 - \frac{y}{k}\right) - h,$$

where r, k and h are positive constants. Suppose $r = 1$, $k = 10$, $h = \frac{9}{10}$ and the initial population at time $t = 0$ is $y_0 > 0$. Find the expression of $y(t)$ that satisfies the given differential equation.

2. (a) Consider the following system of differential equations :

$$\frac{dx}{dt} = -k_1 x, \; x(0) = x_0,$$

$$\frac{dy}{dt} = k_1 x - k_2 y, \; y(0) = 0,$$

where, k_1, $k_2 > 0$ represent the rates at which a drug, such as antihistamine or decongestant, moves between two compartments in the body : the gastrointestinal (GI) tract and the bloodstream. When a patient takes a single pill, $x(t)$ denotes the level of the drug in the GI-tract and $y(t)$ denotes the level in the bloodstream at time t.

(i) Find the expressions for $x(t)$ and $y(t)$ that satisfy the given system of differential equations when $k_1 \neq k_2$.

(iii) Is the solution obtained in Part (a) still valid when $k_1 = k_2$? If not, find the solution in this special case.

(b) Find all equilibrium solutions of the differential equation :

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{k}\right)$$

where r and k are positive constants.

Also, discuss the stability of each equilibrium solution.

(c) Let C(t) denote the concentration of pollutant in a lake at time t, and let the lake have a volume V (in litres). Let F be the constant rate (in m^3/day) at which water flows out of the lake. Assume that water flows into the lake at the same rate F, with pollutant concentration c_{in} (in units of mass per unit volume). The rate of change of the concentration of the pollutant in the lake is governed by the following differential equation :

$$\frac{dC}{dt} = \frac{F}{V}c_{in} - \frac{F}{V}C,$$

where $C(0) = C_0$ is the initial concentration of the pollutant in the lake.

(i) Find an explicit expression for $C(t)$ which satisfies the given differential equation.

(ii) How long will it take for the lake pollution level to reach 5% of its initial level, if only fresh water flows into the lake ?

3. (a) Consider a disease where all the induced contagious infectives recovers. Consider the per unit death and birth rate constant. Develop a system of differential equations for the number of susceptibles $S(t)$, number of infectives $I(t)$ and the number of recovered ones $R(t)$.

(b) In a following model, for predator $Y(t)$-prey $X(t)$ with density dependence for the prey and DDT acting on both species at the rate p_1, p_2 respectively.

$$\frac{dX}{dt} = a_1X\left(1 - \frac{X}{K}\right) - c_1YX - p_1X$$

$$\frac{dY}{dt} = c_2YX - d_2Y - p_2Y$$

(i) Find the equilibrium points.

(ii) Discuss the possibility of extinction of one of the prey or predators.

(c) Consider density dependence in competing species model

$$\frac{dX}{dt} = a_1X - c_1XY - d_1X^2,$$

$$\frac{dY}{dt} = a_2Y - c_2XY - d_2Y^2 \text{ with } X(0) = X_0,\ Y(0) = Y_0$$

where $a_1 = 2$, $a_2 = 3$, $c_1 = 3$, $c_2 = 1$, $d_1 = 2$, $d_2 = 2$.

(i) What is the carrying capacity for each of the species for the given parameter's values ?

(ii) Find the equilibrium points.

4. (a) Consider the following equations for battle model in case of aimed firing between Red army R(t) and Blue army B(t) :

$$\frac{dR}{dt} = -a_1B,$$

$$\frac{dB}{dt} = -a_2R,$$

where $a_1 = 10^{-4}$, $a_2 = 10^{-1}$, B(0) = 100 and R(0) = 1000.

(i) Use the chain rule to find a relationship between the number of soldiers in both the army.

(ii) Deduce the direction of travel along the trajectories, providing reasons.

(b) Consider a population split into two groups : adults and juveniles, where the adults give birth to juveniles but juveniles are not yet fertile. Eventually juveniles mature into adults. You may assume constant per capita birth- and death-rates for the population, and also assume that the young mature into adults at a constant per capita rate b. Starting from assumptions if any more needed then suitable word equations or a compartment diagram, formulate a pair of differential equations describing the density of adults.

(c) The predator Y(t)-prey X(t) equations with additional deaths due to some external factor at a constant rate p_1 and p_2 are

$$\frac{dX}{dt} = a_1 X - c_1 XY - p_1 X \text{ and}$$

$$\frac{dY}{dt} = -a_2 Y + c_2 XY - p_2 Y$$

where all the parameters are positive.

(i) Find the equilibrium points.

(ii) Is there any possibility of extinction of both the population of predator and prey ? Justify.

(*iii*) Show that the predator fraction of the total average population is given by

$$f = \frac{1}{1+\left[\frac{c_1(a_2+p_2)}{c_2(a_1-p_1)}\right]}.$$

5. (*a*) Using Monte-Carlo Simulation, write an algorithm to find the area beneath the curve $y = 2\cos x$ in the interval $0 \le x \le \frac{\pi}{2}$.

(*b*) Use the linear congruence method $x_{n+1} = (ax_n + b) \bmod(c)$ to generate 10 random numbers using $a = 1$, $b = 7$, $c = 10$ and a seed $x_0 = 7$. Is there any drawback to this method ? Justify.

(*c*) Use the transformed least squares method to estimate the value of a in the model $y = ax^3$, given the data points :

x	y
1	5
2	3
3	8
4	7
5	5
6	8

6. (a) Use the middle-square method to generate 10 random numbers using a seed $x_0 = 1009$. Is there any drawback to this method ? Justify.

(b) Use the least-squares method to estimate the coefficients a and b in the linear model $y = ax + b$, given the data points :

x	y
0	3
1	6
2	8
3	11
4	13
5	14

(c) Using Monte-Carlo Simulation, write an algorithm to find the probability of getting a head when tossing an unfair coin, assuming the coin is biased according to the empirical distribution : P(*head*) = 0.35 and P(*tail*) = 0.75.