(c) Use Monte Carlo simulation to approximate the probability of heads occurring when n times a fair coin is tossed.

[This question paper contains 8 printed pages.]

Your Roll No..............

Sr. No. of Question Paper : 5105 **J**

Unique Paper Code : 2353200011

Name of the Paper : Introduction to Mathematical Modeling

Name of the Course : **BA(P)/BSc (Physical Science & Mathematical Science): DSE**

Semester : VI

Duration : 3 Hours Maximum Marks : 90

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. All the **six** questions are compulsory.
3. Attempt any **two** parts from each question.
4. **All** questions carry equal marks.
5. Use of Scientific Calculator is allowed.

1. (a) A population of a certain country grows at a rate of 5% per year. If the initial population is 100, in how much time the population will double.

(b) Suppose two small tributaries viz A and B falls into the river C. There are two sewage systems R1 and R2 that falls into A and contributes towards polluting it. Also the sewage system R3 falls into B and pollutes its water. There is a water cleaning system installed at river C. Create a compartment diagram that captures the flow of pollutant in A, B and C. Also writes the corresponding word equations.

(c) Suppose a lake has constant volume V, which is continuously well mixed. Let C(t) be the concentration of pollutant at time t and F is the flow rate in the lake. Let C_{in} be the concentration of pollutant in the incoming water. Answer the following:

(i) Derive a differential equation that captures the rate of change of C(t).

(ii) Suppose that $C_{in} = 100$, Plot the a representative graphs of C(t), when initial pollutant concentration level in lake equals to 50 and 150.

(b) The following data is given for the variables x and y :

x	0.1	0.2	0.3	0.4	0.5
y	0.06	0.12	0.36	0.65	0.95

Formulate the least-squares mathematical model of the form $y = c_1x^2 + c_2x + c_3$ that minimizes the sum of the squared deviations between data and the model.

(c) The following data is given :

x	1	2	3	4	5
y	1	1	2	2	4

Fit the data using the least-squares method for $y = ax + b$.

6. (a) What is a random number ? Use the middle square method to generate 10 random numbers using 1009 as seed.

(b) Using Monte Carlo simulation, write an algorithm to calculate part of the volume of an ellipsoid $x^2/2 + y^2/2 + z^2/9 \leq 16$ that lies in the first octant $x > 0, \ y > 0, \ z > 0$.

(ii) Find a relation between Y and X by chain rule for the differential equations

$$\frac{dX}{dt} = -XY, \frac{dY}{dt} = -2Y.$$

(c) For the predator-prey model with density dependence for the prey and DDT acting on both species,

$$\frac{dX}{dt} = \beta_1 X\left(1 - \frac{X}{K}\right) - c_1 XY - p_1 X, \quad \frac{dY}{dt} = c_2 XY - \alpha_2 Y - p_2 Y,$$

Show that (X,Y) is one equilibrium point,

where $X = \frac{\alpha_2 + p_2}{c_2}$, $Y = \frac{\beta_1\left(1 - \frac{X_e}{K}\right) - p_1}{c_1}$, and

determine if there are any other equilibrium points.

5. (a) Explain the process of fitting a linear model y = ax + b to data. What are absolute deviations and how are they used to find the best fit? State one limitation of using absolute deviations. What is the minimize approach and how does it differ from minimizing the sum of deviations?

2. (a) Consider the drug assimilation model in which when a pill is taken by an individual it first goes to the gastrointestinal tract (GI-tract), and then diffused in the bloodstream. Let X(t) and Y(t) represents the amount of pill in the GI-Track and blood respectively at time t. Derive differential equations that captures the rate of change of X(t) and Y(t) in the case when single pill is taken by the individual. Further plot a representative graph for X(t) and Y(t) that shows the amount of drug in the GI Track and blood over a long period of time.

(b) Let X(t) denotes the population at time t. By taking suitable parameters derive the Logistic Equation that captures the density dependent growth rate. Identify the parameter K, carrying capacity.

(c) Suppose that the population of a country is modelled using the differential equation $\frac{dX(t)}{dt} =$ $0.2\,X - 0.001\,X^2$, with initial population equals to 100. The unit of time is in month. Find the population of the country after 2 months.

3. (a) When considering a disease,

(i) make some assumptions to construct a compartmental diagram for the model and develop appropriate word equations for the rate of change to susceptible and contagious infectives

(ii) Using the assumptions and the word equation (i) and formulate differential equations.

(b) Let $X(t)$ denote the number of prey per unit area and $Y(t)$ the number of predators per unit area.

(i) Make some assumptions to formulate differential equations for the prey and predator densities.

(ii) Check the Lotka-Volterra model in the limiting cases of prey with no predators or predators with no prey.

(c) Determine the appropriate input-output diagram and associated differential equations for the number of soldiers in both the red and blue armies, and

(i) modify the model to account for one, or both, of the armies using random fire.

(ii) modify the model to account for loss due to disease and / or gains due to reinforcements.

4. (a) When considering ecosystem, where two populations compete for limited resources such as food or territory.

(i) make some assumptions and construct compartmental diagram for the competing species and develop appropriate word equations for each of the populations.

(ii) formulate differential equations for assumptions and word equation (i).

(b) (i) Find all the equilibrium solutions of the equations

$$\frac{dX}{dt} = 2X - XY, \frac{dY}{dt} = Y - XY.$$