(c) Prove that in a ring R with unity 1,

(i) if 1 has infinite order under addition, then the characteristic of R is 0,

(ii) if 1 has finite order n under addition, then the characteristic of R is n.

Hence or otherwise state the characteristics of the rings $\mathbb{Z}$, $M_2(\mathbb{Z})$ and $\mathbb{Z}_3[i]$.

6. (a) State the ideal test. Let R be a commutative ring with unity and let $a \in R$. Then verify that the sets $I = \{ra \mid r \in R\}$ and $S = \{ar \mid r \in R\}$ are ideals of R.

(b) Let $\phi: R \to S$ be a ring homomorphism. Prove that :

(i) $\phi(nr) = n\phi(r)$ and $\phi(r^n) = (\phi(r))^n$ for any $r \in R$ and any positive integer n.

(ii) $\phi(R)$ is commutative if R is commutative.

(c) Prove that every ring homomorphism ϕ from $\mathbb{Z}_n$ to itself has the form $\phi(x) = ax$, where $a^2 = a$. Also, find $\operatorname{Ker}\phi$.

[This question paper contains 4 printed pages.]

Your Roll No..............

Sr. No. of Question Paper : 5106 **J**

Unique Paper Code : 2354000009

Name of the Paper : Abstract Algebra

Name of the Course : **COMMON PROG GROUP**

Semester : VI

Duration : 3 Hours Maximum Marks : 90

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. All the **six** questions are compulsory.
3. Attempt any **two** parts from each question.
4. Each part carries **7.5** marks.
5. Use of Calculator not allowed.

1. (a) Define $SL(2, \mathbb{F})$ and find the inverse of the element $\begin{bmatrix} 3 & 4 \\ 4 & 4 \end{bmatrix}$ in $SL(2, \mathbb{Z}_5)$.

(b) Describe the symmetries of a square. Construct the corresponding Cayley Table.

(c) Prove that the set $G = \left\{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \mid a, b, c \in \mathbb{R} \right\}$ is a group under matrix multiplication.

2. (a) If H and K are subgroups of a group G, show that $H \cap K$ is a subgroup of G. What can you say about $H \cup K$? Justify.

(b) Let G be a group. Then prove that: $c(a) = c(a^{-1})$ $\forall$ a $\in$ G, where c(a) is the centralizer of α in G.

(c) Prove that the center of a group G is a subgroup of G. Find the center of the group D_4. Justify.

3. (a) Define a cyclic group. Find an example of a noncyclic group, all of whose proper subgroups are cyclic. Justify your answer.

(b) Let $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 4 & 5 & 1 & 7 & 8 & 6 \end{pmatrix}$,

$\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 3 & 8 & 7 & 6 & 5 & 2 & 4 \end{pmatrix}$.

Write α, β and $\alpha\beta$ as product of disjoint cycle and also find β^{-1}.

(c) State the Lagrange's Theorem. Let $|G| = 60$, then what are the possible orders for subgroups of G.

4. (a) Let H be a subgroup of a group G, then show that either $\alpha H = bH$ or $\alpha H \cap bH = \phi$ for every α, b $\in$ G. Also, find all the left cosets of {1,11} in U(30).

(b) Define the factor group. Also, let $G = \mathbb{Z}_{24}$, then find the order of the element $14+\langle 8\rangle$ in the factor group $\mathbb{Z}_{24}/\langle 8\rangle$.

(c) Consider the map $f: \mathbb{Z} \to \mathbb{Z}_n$ defined by f(m) = mmodn. Show that f is a homomorphism. Also, find the Kerf.

5. (a) Show that the set $\mathbb{Z}[x]$ of all polynomials in the variable x with integer coefficients under addition and multiplication is a commutative ring with unity f(x) = 1.

(b) State the subring test. Prove that in a ring R, the set $\{x \in R \mid \alpha x = x\alpha \text{ for all } \alpha \in R\}$ is a subring of R. Find all the subrings of the ring $\mathbb{Z}$.