This question paper contains 3 printed pages]

Roll No. | | | | | | | | | | | |

S. No. of Question Paper : **7439**

Unique Paper Code : **2352572401**

Name of the Paper : **Abstract Algebra**

Name of the Course : **B.A./B.Sc. (Programme) Mathematics as Non-Major/Minor-DSC**

Semester : **IV**

Duration : **3 Hours** Maximum Marks : **90**

(Write your Roll No. on the top immediately on receipt of this question paper.)

Attempt *all* questions by selecting any *two* parts from each question.

Each part carries **7.5** marks.

Use of calculator is not allowed.

1. (*a*) Let a and n be positive integers and let $d = gcd(a, n)$. Prove that the equation $ax \bmod n = 1$ has a solution if and only if $d = 1$.

   (*b*) Define Group. Show that the general linear group of 2×2 matrices over $\mathbf{R}$ is a group.

   (*c*) Prove that a group G of order 3 must be cyclic.

2. (*a*) Define Abelian group. Show that a group G is Abelian if and only if $(ab)^{-1} = a^{-1}b^{-1}$ for all a, b in G.

(*b*) Define order of an element in a group. Find the order of each element of U(15).

(*c*) State and prove one-step subgroup test. Use it to prove that if G is an abelian group with identity e, then $H = \{x \in G \mid x^2 = e\}$ is a subgroup of G.

3. (*a*) If $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 4 & 6 & 2 & 5 \end{pmatrix}$ and $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 6 & 3 & 5 & 1 & 2 \end{pmatrix}$, then :

(*i*) Find $O(\alpha\beta)$

(*ii*) Find $O(\beta\alpha)$

(*iii*) Find α^{-1}.

(*b*) Find all the right cosets of {1, 15} in U (32).

(*c*) Show that the factor group $\mathbf{Z}/5\mathbf{Z}$ is a cyclic group of order 5.

4. (*a*) Define normal subgroup. Prove that SL(2, **R**), the group of 2 × 2 matrices with determinant 1 is normal subgroup of GL(2, **R**), the group of 2 × 2 matrices with non-zero determinant.

(*b*) Show that $\phi : Z_{12} \to Z_{12}$ given by $\phi(x) = 4x$ is a group homomorphism.

(*c*) Show that the kernel of homomorphism $\phi : \mathbf{R}^* \to \mathbf{R}^*$ given by $\phi(x) = |x|$ is a cyclic group.

5. (a) Prove that Z_4 is a ring and find its set of units.

(b) Let x, y and z belong to a ring R. Then show that :

(i) $(-x)(-y) = xy$

(ii) $x(y - z) = xy - xz$.

(c) State ideal test and prove that $2\mathbf{Z} = \{2n \mid n \in \mathbf{Z}\}$ is an ideal of $\mathbf{Z}$.

6. (a) Define zero-divisors and integral domain. Give an example of a commutative ring without zero-divisors that is not an integral domain.

(b) Let a, b and c belong to an integral domain. If $a \neq 0$ and $ab = ac$, then show that $b = c$.

(c) Let f be a ring homomorphism from a ring R to a ring S and let A be a subring of R. Then prove that $f(A) = \{f(a) \mid a \in A\}$ is a subring of S.